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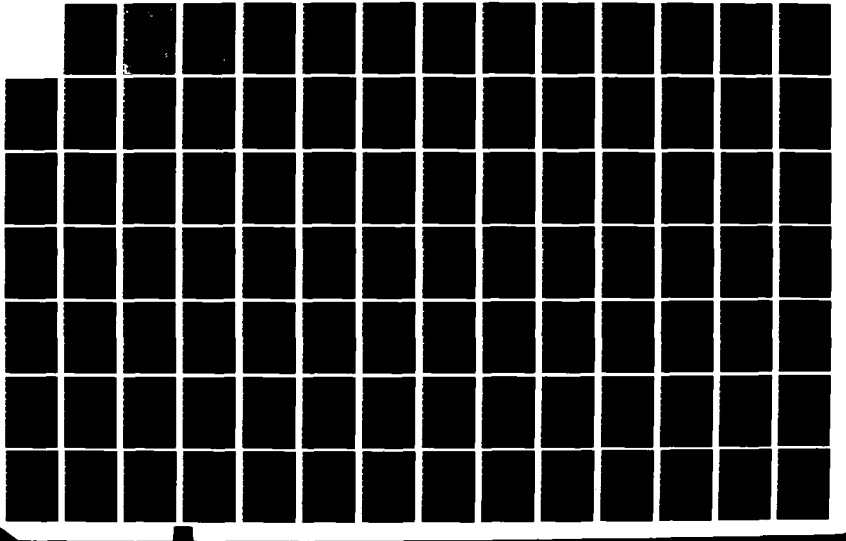

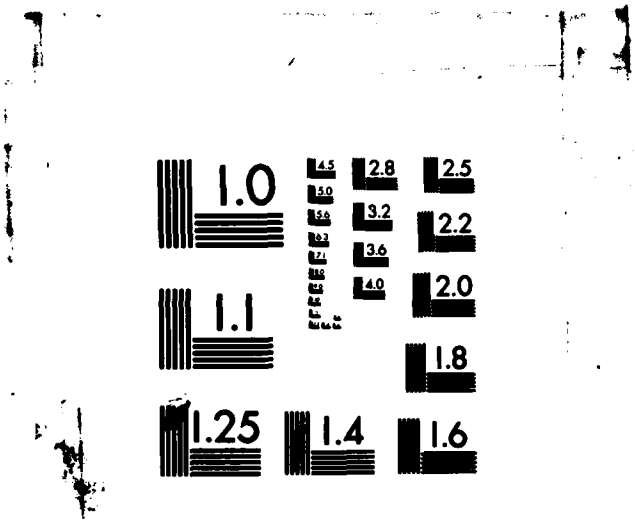

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VARIANCE REDUCTION TECHNIQUES

WITH APPLICATIONS

THESIS

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Lieutenant Colonel, Egyptian Army

AFIT/MA/GOR/83D-2

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THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Operations Research

Mohamed Refat Elhefny

Lieutenant Colonel, Egyptian Army

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Preface

The purpose of this study was to survey and use variance reduction techniques. To achieve this, extensive literature search was done. For each technique, the basic idea, analytical formulation, method of implementation, fields of application, advantages, disadvantages and any other specific characteristics were identified, presented and clarified. To illustrate each technique, all the above characteristics were tabulated. Numerical examples with computer programs were given and, finally, examples of application of the most commonly used techniques were presented.

This work should result in better understanding of variance reduction techniques so that one can use them more efficiently.

I would like to thank my thesis advisor, Professor Albert H. Moore, and my reader, Major Joseph W. Coleman, for their continuous patience and assistance. I also wish to thank Ms. Sharon Gabriel for her excellent typing of this thesis.

Mohamed Refat Elhefny

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Abstract

The objective of this study was to find out about, analyze and illustrate the characteristics of variance reduction techniques. Extensive literature review was done to identify the basic idea, theoretical foundation, procedure for implementation, fields of application, and other specific characteristics of each technique.

Examples were given to show how to implement the commonly used techniques. Computer programs were written to perform those examples. Results were used to compare the efficiency of different techniques. Three studies in the fields of inventory, queuing and computer performance measurement were discussed where different variance reduction technqiues were employed. Conclusions and recommendations were given.

VARIANCE REDUCTION TECHNIQUES WITH APPLICATIONS

I. Introduction

During the early days of simulation (1940-1960), when computer speeds were much slower, investigators found themselves in a position where it was very expensive to decrease the variation of estimates by increasing the sample size. Consequently, interest grew in developing sample-estimating procedures that could either increase the precision of estimates for a fixed sample size or, conversely, decrease the sample size required to obtain a fixed degree of precision. Those procedures are often referred to as Variance Reduction Techniques (VRTs). The underlying principle in those procedures is the utilization of knowledge about the structure of the model and properties of the input data to change or distort the original problem so that special techniques can be used to obtain the desired estimates at a lower cost.

Historically, most of the underlying statistical approaches used in VRTs had been in use much earlier for different purposes, but during the period of 1940-1960 the techniques had been refined for specific use as variance reduction techniques in computer simulation. As the computer speeds increased, the interest in those techniques

declined. However, the recent increase in complexity of computer simulation, due to handling complex models for large problems, renewed the interest in the use and development of variance reduction techniques. In some models of complex systems, obtaining a single sample may require a great deal of computer time when a high speed computer is used. In such cases, the use of variance reduction techniques is vital.

Kahn was one of the first pioneers who clarified most of the techniques. He explained and illustrated most of them in the report published by Rand in 1956 (19). He presented several examples pertaining to the area of radiation transport to demonstrate the applicability of VRTs. Hammersley and Handscomb presented the general Monte Carlo concepts and methods (11). The most comprehensive overview of the use of VRTs is presented in their book and also in the book by Spanier and Gelbard (30), where standard variance reduction techniques, along with several applications to radiation transport problems, are discussed. Other books (16; 28) give less rigorous summaries of VRTs which are helpful for the understanding of the basic ideas behind each technique. On the other hand, many articles (7; 22; 23), reports (6; 8; 12; 13), and studies (5; 10) have been devoted to development and application o ~er' an VRTs for a specific kind of problem.

To compare different variance reduction techniques when they are applicable to a certain problem, efficiency of simulation in estimating parameters is used. It is first suggested by Hammersley and Handscomb (11) and is defined as

$$\text{Efficiency} = \frac{1}{\text{variance X work}}$$

They also defined the relative efficiency of simulation when applying two Monte Carlo techniques as the ratio of their efficiencies.

This implies that a reduction in variance of estimator is not worthwhile if the work required to achieve it is excessive. Therefore, one should take into consideration the cost or the work required to achieve the anticipated variance reduction. In reality, one cannot estimate the required work or the potential variance reduction for a given method. The analyst can use his experience and intuition to choose the suitable method to solve his problem. In some cases, the use of any of the techniques is infeasible or improfitable, but if applicable and properly used, VRTs can provide a tremendous increase in the efficiency of the simulation.

Shannon (26) stated that variance reduction techniques are not new, but they are not widely practiced in spite of

the great saving of work or reduction of variance which can be achieved when suitable VRTs are applied to certain cases.

The lack of attention given to those techniques is a consequence of the shortage of text books discussing them and the inconvenience to analysts when using them. This effort is devoted to treat those problems with the hope of making VRTs more convenient to use.

In the next chapter, each of the known variance reduction techniques is illustrated. The types of problems which can be handled by the techniques are discussed.

In the third chapter, selection and implementation of VRTs are discussed in detail, and procedure for selection of the suitable VRT to a certain type of problem is illustrated. A lengthy table of the characteristics of all the available standards is given where the description, criteria of application, advantages, disadvantages and fields of application of each of the techniques are condensed to help in the selection of the suitable technique.

In Chapter Four, steps for implementation of each technique are given in a simple form including formulae for calculating the estimator and the variance of the estimation. Simple examples for most of the standard

techniques are used to demonstrate the implementation of the techniques.

In the fifth chapter, selected examples of real applications of various VRTs are given. Applications in the fields of inventory control simulation models (6), queuing simulation models (10), and computer performance measurements (7) are demonstrated where the most commonly used VRTs are applied.

II. Variance Reduction Techniques

Classification of Variance Reduction Techniques

All VRTs are concerned with increasing the accuracy of Monte Carlo estimates of parameters at a fixed sample size or decreasing the sample size required to achieve a certain degree of accuracy. In general, VRTs aim to improve the efficiency of the simulation process when contrasted with crude (direct or straightforward) Monte Carlo simulation which attempts to create true-to-life or actual modeling for the underlying process. In crude Monte Carlo simulation, random sampling, flows through the model and sampling probability distributions are chosen to reflect the real situation as exactly as possible. On the other hand, VRTs attempt to increase the effectiveness of Monte Carlo simulation by one of the following approaches:

1. Modifying the sampling process
2. Utilization of approximate or analytical information
3. Studying the system within a different context.

According to these approaches, the known VRTs can be classified though many of them are closely related, which makes it difficult to completely classify them. The suggested classification of most of the known VRTs is given as follows (24):

Modification of the Sampling Process

- Importance Sampling
- Russian Roulette and Splitting
- Systematic Sampling
- Stratified Sampling

Use of Analytical Equivalence

- Expected Values
- Statistical Estimation
- Correlated Sampling
- History Reanalysis
- Control Variates
- Antithetic Variates
- Regression

Specialized Techniques

- Sequential Sampling
- Adjoint Formulation
- Transformation
- Orthonormal Functions
- Conditional Monte Carlo

Modifying the sampling process is usually achieved by using more effective sampling techniques or altering the sampling distributions. This approach is beneficial, if not necessary, to handle simulation problems involving very low probability events. In such a case, a modified sampling scheme is required to increase the number of occurrences of these rare events.

Using the analytical equivalence is another approach for reducing the variance of estimation in the simulation process. Since analytical procedures, if available, are usually preferable to simulation, one should replace the results obtained through simulation at any part of the

process by the available analytical results or estimates.

In addition to sampling modification and analytical equivalence, there are certain specialized techniques that can be used to achieve variance reduction. These procedures may include the application of one or more of the above techniques.

Presentation of Variance Reduction Techniques

Modification of the Sampling Process. Techniques under this class have several common characteristics in that they all reduce the variance of estimate by sampling from a probability distribution different from the true physical one. This will help by observing events of interest more often and hence decreasing the computing time and effort. These techniques also preserve the actual process of system in the simulation model, while only the probability associated with each event is altered.

Importance Sampling

Concepts of the Technique. In this method, sampling is forced to concentrate in the more important regions. In other words, probabilities of occurrence of events are biased in a known fashion so that the resulting bias can be adjusted when interpreting the results.

The idea can be simply illustrated by considering tossing a pair of dice. If one is interested in the occurrence of three as a sum of the two top faces, one

could bias each die toward the numbers one and two in a known fashion. The computation of the results should be altered according to the information from the biasing scheme to unbias the answers.

Mathematically, the importance sampling can be illustrated by considering the Monte Carlo estimate of parameter I where

$$I = E[g(x)] = \int g(x)\, f(x)\, dx \tag{2.1}$$

The crude Monte Carlo procedure for estimating I would be as follows:

1. Select a random sample $x_1, \ldots, x_N$ from the distribution with density function $f(x)$

2. Estimate I using

$$I = \frac{1}{N} \sum_{i=1}^{N} g(x_i) \tag{2.2}$$

The sample variance of this estimate is given by

$$S^2 = \frac{N}{N-1} \left[\frac{1}{N} \sum_{i=1}^{N} g^2(x_i) - \hat{I}^2\right] \tag{2.3}$$

Considering another distribution $f^*(x)$, one can write

$$I = \int \frac{g(x) f(x)}{f^*(x)}\, f^*(x) dx \tag{2.4}$$

where $f^*(x) \neq 0$ when $g(x)f(x) \neq 0$. If we sample from $f^*(x)$ taking $x_1,..,x_N$ randomly, a new estimator $\hat{I}_1$ can be calculated as

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} \frac{g(x_i)f(x_i)}{f^*(x_i)} \tag{2.5}$$

Each sample should, then, be weighed by $\frac{f(x_i)}{f^*(x_i)}$ in the final result. This variance reduction procedure will lead to a sample variance given by

$$S_1^2 = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} [\frac{g(x_i)f(x_i)}{f^*(x_i)}]^2 - \hat{I}_1^2\} \tag{2.6}$$

Consider the expected value $E[(\hat{I}_1 - I)^2]$ that is

$$E[(\hat{I}_1 - I)^2] = \frac{1}{N} [(\frac{1}{N} \sum_{i=1}^{N} \frac{g(x_i)f(x_i)}{f^*(x_i)} - I)^2] \tag{2.7}$$

If $f^*(x) = \frac{g(x)f(x)}{I}$ and $f^*(x)$ is non-negative, then $E[(I_1 - I)^2] = 0$ which is a desirable situation that can be achieved only if I is known and $f^*(x)$ is chosen to be equal to $\frac{g(x)f(x)}{I}$. Since I is always unknown, one has to use available information about the problem to choose $f^*(x)$ as close to $\frac{g(x)f(x)}{I}$ as possible. If one fails to choose a suitable $f^*(x)$ to sample from, importance sampling can give a worse result than that of

crude Monte Carlo; that is, when

$$E[S^2 - S_1^2] = \int g^2(x)[1 - \frac{f(x)}{f^*(x)}] f(x)dx$$

is not positive.

Importance Sampling for More than One Variable. If the functions f and g are functions of a vector of random variables $\vec{X}$, one can take a random sample $\vec{X}_1, \vec{X}_2, \ldots \vec{X}_N$ from a selected probability function $f^*(\vec{X})$ and estimate the parameter I as

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} \frac{g(\vec{X}_i) f(\vec{X}_i)}{f(\vec{X}_i)} \tag{2.8}$$

with the variance

$$S_1^2 = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} [\frac{g(\vec{X}_i) f(\vec{X}_i)}{f^*(\vec{X}_i)}]^2 - \hat{I}_1^2\} \tag{2.9}$$

Also in this case, if we select $f^*(\vec{x}) = \frac{g(\vec{x}) f(\vec{x})}{I}$, then $E[(\hat{I}_1 - I)^2]$ will be zero indicating the best selection of $f^*(\vec{x})$ which cannot be done without knowing I, the parameter to be estimated. Due to multidimensionality of the function $f(\vec{x})$, it is difficult to develop $f^*(\vec{x})$ efficiently. Therefore, a conditional importance function may be selected instead of $f^*(\vec{X})$. In the simplest case,

when only two variables x,y are in the function f,g, the parameter I can be estimated as

$$I = \int_{x,y} g(x,y)f(x,y)dxdy = \int_{x,y} g(x,y)f(x)f(y/x)dxdy$$

$$= \int_{x,y} \frac{g(x,y)f(x)}{f^*(x)} f^*(x)f(y/x)dxdy \qquad (2.10)$$

So one can select $X_1,X_2,\ldots,X_N$ randomly from a function $f^*(x)$ and $Y_1,Y_2,\ldots,Y_N$ from $f(y/X_i)f^*(X_i)$ and estimate I as

$$\hat{I}_1 = \frac{1}{N} \Sigma \frac{g(X_i,Y_i)f(X_i)}{f^*(X_i)} \qquad (2.11)$$

The sample variance will be

$$S_1^2 = \frac{N}{N-1} \{[\frac{g(X_i,Y_i)f(X_i)}{f^*(X_i)}]^2 - I_1^2\} \qquad (2.12)$$

In this case

$$E[\hat{I}_1 - I)^2] = \frac{1}{N} \{\int_x \frac{f^2(x)}{f^*(x)} \int_y g^2(x,y)f(y/x)dydx - I^2\} \qquad (2.13)$$

Using the relation

$$E[g^2(x,y)/x] = \int_y g^2(x,y) f(y/x)\,dy \tag{2.14}$$

one can specify the best importance function to be

$$f^*(x) = \frac{f(x)\{E[g^2(x,y)/x)^{\frac{1}{2}}}{\int\{E[g^2(x,y)/x]\}^{\frac{1}{2}} f(x)\,dx} \tag{2.15}$$

which will reduce $E[(\hat{I}_1 - I)^2]$ to be

$$E[(\hat{I}_1 - I)^2] = \frac{1}{N}\{\int\{E[g^2(x,y)/x)\}^{\frac{1}{2}} f(x)dx]^2\} \tag{2.16}$$

If we have more than two variables, the best importance function can be expressed as

$$f^*(x) = \frac{f(x)\{E[g^2(x,\vec{y})/x]\}^{\frac{1}{2}}}{\int\{E[g^2(x,\vec{y})/x]\}^{\frac{1}{2}} f(x)\,dx} \tag{2.17}$$

The vector $\vec{y}$ stands for all the random variables except x . The estimate of I and the variance of sample are expressed in a way similar to that of the two dimensional case. It should be noticed that the selection of the best $f^*(x)$ can be done only if we know the estimator I for which the whole simulation is made, so one can only select a good $f^*(x)$ guided by the given formula for best $f^*(x)$.

Russian Roulette and Splitting

Concepts of the Technique. Von Neumann and Ulan first used these two techniques in particle diffusion problems. The combination of the two techniques can be extremely effective when some knowledge about the importance of the regions of the distribution is available. If the problem is structured as a series of events that can be examined at various stages, at some of these stages, one can tell whether a process would contribute to the desired result or not. If the state of a certain stage is not of interest, the process will be killed off with a known probability. This is called Russian Roulette. On the other hand, if the process is an interesting state, additional investigation might be conducted by increasing the number of simulations starting from that situation. This is called Splitting. As mentioned, the combined technique can be very effective in multistage problems such as random walk, subsystems in series, etc. It could also be useful in simulations involving a large number of discrete situations such as queuing systems in which large numbers of individuals are being tracked. In such systems, Russian Roulette can be used to decrease the number of individuals being tracked by removing an individual at a certain stage in the problem with probability P_i . Otherwise, that individual is allowed to continue in the system with a new

weight $W = (1 - P_i)^{-1} = 1/q_i$ (24). This can be repeated for the other individuals and so the number of individuals in the system can be decreased. Splitting can increase the number of tracked individuals in the system by replacing an individual with weight W by n individuals; each of them has the weight W/n. Those individuals can independently proceed through the system keeping their assigned weights.

Application of the Technique to a Two-Stage Problem. In a two-stage proble, if X is the random observation from the first stage and Y is that from the second, the estimation of $I = E[g(x,y)]$ can be calculated as

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} g(X_i, Y_i) \qquad (2.18)$$

where a sample of pairs of values, (X_1,Y_2), (X_2,Y_2), ... , (X_N, Y_N) are generated from the given distribution of x and y. If some values of X would lead to more interesting results than others, one can use Russian Roulette and Splitting to divide the states in the first stage into the two following sets:

S_1 : The set of states which will be terminated by Russian Roulette with probability $P = 1-q$, but if the

simulation is continued for a state, the estimated parameter will by weighted by $1/q$.

S_2 : The set of states which will continue in the simulation. Each will be split into n simulations with weight = $\frac{W}{n}$ for each.

The procedure would be repeated for N starting simulations and the modified estimator will be

$$\hat{I} = \frac{1}{N} \{ \sum_{X_i \epsilon S_1} \frac{g(X_i,Y_i)}{q} + \sum_{X_i \epsilon S_2} \sum_{j=1}^{n} \frac{g(X_i,Y_i)}{n} \} \tag{2.19}$$

which is the unbiased estimator for I (24).

The sample variance in this case is given by

$$S^2 = \frac{N}{N-1} \{ \frac{1}{N} \sum_{i=1}^{N} I_i^2 - \hat{I}^2 \} \tag{2.20}$$

where $I_i = 0$, $g(X_i,Y_i)/q$ or $\sum_{j=1}^{n} g(X_i,Y_i)$ according to contribution to the estimator from the history i , and

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} I_i \tag{2.21}$$

<u>Weight Standard for General Application of the Technique</u>. If the problem to be simulated is broken into N regions, two weights W_{H_i} and W_{L_i} will be

assigned to each region i . When a history enters region i , its current weight W will be compared to the region weight standard using the following rules (24):

1. If $W < W_{L_i}$, Russian Roulette is applied as follows:

   * kill the history with $P = 1 - \frac{W}{W_{A_i}}$
   * the history will survive with $P = \frac{W}{W_{A_i}}$ and its new weight will be W_{A_i}

2. If $W > W_{H_i}$, Splitting is applied as follows:

   * find n such that $W - n\, W_{A_i} < W_{A_i}$
   * create n histories which starts from this point, each with weight W_{A_i}
   * with probability $\frac{W - n\, W_{A_i}}{W_{A_i}}$, create one more history starting from the same point with weight W_{A_i}

3. If $W_{L_i} < W < W_{H_i}$, let the history continue in the simulation without any change.

The above procedure will be used under the assumption of approximately constant importance for each region. The importance of a region is inversely proportional to its average weight W_{A_i} . This means that histories moving

into a region of higher importance (lower weight) will be split, while those moving to a region of lower importance (higher weight) will suffer Russian Roulette. To increase the efficiency of computer time utilization, a fixed weight should be used for all histories in a region of constant importance. The high and low weight standards, W_H , W_L , are used only to define the upper and lower limits for triggering Russian Roulette and Splitting processes. They should be used only when another VRT is used besides Russian Roulette and Splitting (24).

Selection of the Suitable Criteria. There are three parameters from which one should choose: weight standards, probability of kill and number of splitted histories. The best selection is the one which minimizes the variance in estimate. It is difficult to perform this optimum selection so the results from importance sampling analysis can help where the weight standards for a given region will be proportional to $(E[g^2(x)])^{-\frac{1}{2}}$ which means that the weight standards should be high in regions of low value and low in regions of high value.

Systematic Sampling

Concept of the Technique. Systematic sampling is a structured modification of sampling procedure to reduce the variance of estimation of the parameter. This

technique always results in variance reduction without involving any significant additional effort, so it should be applied when that is possible, in spite of the small improvement it provides. The technique is applicable to any Monte Carlo problem which has a probability distribution to characterize the initial conditions. There are two methods to implement systematic sampling in Monte Carlo technique to estimate the parameter $I = \int_{-\infty}^{\infty} g(x)f(x)dx$ with a reduced variance.

Method I. The range of the density function $f(x)$ is divided into N regions with equal areas; each equals $\frac{1}{N}$ where N is chosen between 5 and 50. It is clear that

$$\frac{1}{N} = \int_{X \epsilon L_i} f(x)dx \ , \ i=1, \ \ldots, \ N \qquad (2.22)$$

where L_i is the length of the ith interval (region). If a sample of random numbers $R_1, \ \ldots \ , R_n$ is selected from $U(0,1)$, the following sequence of numbers will be generated:

$$R_{ij} = \frac{R_i}{N} + \frac{(j-1)}{N} \ , \ i=1,2,\ldots,n \ , \ j=1,\ldots,N$$

For each value of i , this procedure assigns a value R_{ij} to each interval j , then a corresponding value of

the random variable X_{ij} is determined from

$$R_{ij} = \int_{-\infty}^{X_{ij}} f(x)dx \ , \ i=1,\ldots,n \ , \ j=1,\ldots,N$$

The parameter I can then be estimated as

$$\hat{I} = \frac{1}{n} \sum_{i=1}^{n} \hat{I}_i \tag{2.23}$$

where

$$\hat{I}_i = \frac{1}{N} \sum_{j=1}^{N} g(X_{ij}) \ , \ i = 1,\cdots,n \tag{2.24}$$

The sample variance in this case is

$$S^2 = \frac{n}{n-1} \{\frac{1}{n} \sum_{i=1}^{\hat{}} \hat{I}_i^2 - \hat{I}^2\} \tag{2.25}$$

Notice that in this case we generate only n random numbers from $U(0,1)$ and then generate n X N realizations on the range of $f(x)$.

Method II. This method is generally better than the first one to perform systematic sampling. In this case n independent samples are allocated to each of the regions. This is done by selecting R_{ij} , $i=1,\ldots,n$, $j=1,\ldots,N$ from $U(0,1)$, then n random numbers are allocated to each of the N regions using the relation

$$\check{R}_{ij} = \frac{j - R_{ij}}{N} \quad , \quad i=1,\ldots,n \quad , \quad j=1,\ldots,N \tag{2.26}$$

and the corresponding realization X_{ij} is determined from

$$\check{R}_{ij} = \int_{-\infty}^{X_{ij}} f(x)dx \tag{2.27}$$

The estimator $\hat{I}$ and the variance of the sample, S^2 , are determined using the same formulae, Eqs. 2.23 and 2.25.

The second method will always give better variance reduction, although it requires larger number of random samples from $U(0,1)$. In the two cases, the variance reduction is approximately proportional to N^2 .

Stratified Sampling

Stratified sampling is similar to systematic sampling, but better efficiency is achieved by taking more samples from the region of larger variance. It is a way to combine the features of systematic sampling with those of importance sampling. It can be considered as a special case of systematic sampling where the optimum distribution of samples among the regions is attempted. Usually, systematic sampling and stratified sampling can handle the same type of problems, but the latter is recommended when additional information is available about region contributions to the total variance. In that case, additional reduction in the variance can be achieved.

Structure of Stratified Sampling Scheme.

The range of $f(x)$ is broken up into N regions of length $L_1, L_2, \ldots, L_N$, respectively. The length of the jth region is selected in accordance with a specified probability

$$P_j = \int_{X \varepsilon L_j} f(x)dx \ , \ j=1,\ldots,N \tag{2.28}$$

Notice that, if $P_j = \frac{1}{N}$, $j=1,\ldots,N$, the same sampling structure of systematic sampling will be obtained where N regions of equal areas are used. The rule to select P_j is to select such that the variance in $g(x)f(x)$ is the same in each interval. After determining the lengths of the intervals, the numbers of samples from each interval n_j, $j=1,\ldots,N$ should be determined. If the total number of samples is n where

$$n = \sum_{j=1}^{N} n_j \tag{2.29}$$

An unbiased (24) estimate for I is

$$\hat{I} = \sum_{j=1}^{N} \frac{P_j}{n_j} \left[\sum_{i=1}^{n_j} g(X_{ij}) \right]$$

$$= \sum_{j=1}^{N} P_j \hat{I}_j \tag{2.30}$$

where

$$\hat{I}_j = \frac{1}{n_j} \sum_{i=1}^{n_j} g(X_{ij}) \tag{2.31}$$

The sample variance in this case can be estimated as

$$S^2 = \sum_{j=1}^{N} \frac{P_j^2}{n_j - 1} \sum_{i=1}^{n_j} [g(X_{ij}) - \hat{I}_j]^2$$

$$= \sum_{j=1}^{N} \frac{n_j P_j^2}{n_j - 1} \left[\frac{1}{n_j} \sum_{i=1}^{n_j} g^2(X_{ij}) - \hat{I}_j^2\right] \tag{2.32}$$

As in the case of systematic sampling, the stratified sampling, when compared with crude Monte Carlo, has an efficiency proportional to N^2 .

Selection of the optimum number of samples from each internal n_j is a difficult task. Consider

$$E[(\hat{I}-I)^2] = E[(\sum_{j=1}^{N} P_j \hat{I}_j - I)^2]$$

$$= \sum_{j=1}^{N} \frac{P_j^2 \sigma_i^2}{n_j} \tag{2.33}$$

where σ_i^2 is the variance in the j interval

$$\sigma_j^2 = \int_{X_j \epsilon L_j} \frac{f(x)}{P_j}[g(x) - I_j]^2 dx$$

$$= n_j \, E[(\hat{I}_j - I_j)^2] \qquad (2.34)$$

If n_j's are selected to minimize (2.33) subjected to (2.34), the n_j's should be selected to satisfy

$$n_j \simeq \frac{n \, P_j \, \sigma_j}{\sum_{j=1}^{N} P_j \, \sigma_j} \qquad (2.35)$$

Notice that σ_j's are not known, but they can be estimated using

$$S^2 = \frac{1}{n_j^{\wedge}-1} \sum_{i=1}^{n_j^{\wedge}} [g(X_{ij}) - \hat{I}_j]^2$$

$$= \frac{n_j^{\wedge}}{n_j^{\wedge}-1} \left[\frac{1}{n_j} \sum_{i=1}^{n_j^{\wedge}} g^2(X_{ij}) - \hat{I}_j^2 \right] \qquad (2.36)$$

where $n_j^{\wedge}$ samples are arbitrarily selected in each interval. An iterative scheme can be structured to estimate n_j .

Analytical Equivalence Technique

VRTs in this group are based on using prior knowledge of the processes involved to form analytical or approximate

solutions to the problem being simulated. This can also mean that, if one can find a related process which can be exactly solved using analytical or other low variance techniques, he can derive the difference between the exact and related processes using Monte Carlo technique. Many of the techniques under this group are very closely related in the principles and ideas involved.

Use of Expected Value. This method is based on the fact that an analytical determination of parameter estimator is usually preferred to the results of simulation procedures. Since Monte Carlo estimation of a parameter is an estimation of its expected value, the technique is so called and it is applicable where the expected value of portions of the model can be determined analytically without losing an essential element of the simulation. Expected value method can be used in multistage problems where the expected value of the parameter(s) can be analytically determined in one stage or more. For example, consider the two stage problem where X is selected from $f(x)$ at the first stage and Y is selected from $f(y/x)$ at the second. Repeating the process N times, crude Monte Carlo estimation of I is

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} g(X_i, Y_i) \qquad (2.37)$$

If it is possible to determine analytically $f(y/x)$ in the second stage for a given value X from the first stage, the simulation process can be simplified where N samples from $f(x)$, $X_1,\ldots,X_N$ are generated and the estimator for I can be calculated as

$$\hat{I}_E = \frac{1}{N} \sum_{i=1}^{N} E[g(y/X_i)] \tag{2.38}$$

which is an unbiased estimator of I since $E[I_E] = I$.

The sample variance in this case is given by

$$S^2 = \frac{N}{N-1} \left[\frac{1}{N} \sum_{i=1}^{N} E^2[g(y/X_i - \hat{I}_E^2\right] \tag{2.39}$$

In addition to the simplification of the simulation processes, the above technique always gives better results compared with crude Monte Carlo.

It should be noticed that it is not always possible to calculate the expected value of a portion of the simulation analytically. An approximation of the expected value obtained by another variance reduction technique may be used. In some cases, a portion of simulation cannot be replaced by its expected value even if it is analytically determined. In those cases, the second and higher moments may be important in the simulation procedures and not only the expected value.

Statistical Estimation. In this technique, the stochastic process is not removed from the simulation, but the expected value, rather than the simulation result, is used in the estimation.

If one step in a simulation is a random choice between reaching some final outcome or continuing in the simulation process, then Statistical Estimation can be used. In crude Monte Carlo, a random number R would be generated at this step and if $R < P(Y_f/\vec{X})$, then the history would be terminated with score 1. If $R > P(Y_f/\vec{X})$, then the history would continue with no score being made. After N histories, the estimate for probability of reaching Y_f would be

$$\hat{P}_c(Y_f) = \frac{n}{N} \tag{2.40}$$

where n is the number of histories terminated at Y_f.

In statistical estimation, the same simulation process is used, but the estimation technique is changed. Every time the particular step is encountered, a contribution of $P(Y_f/X)$ is added to the estimate, regardless of the actual outcome of the simulation. The final estimate is then given by

$$\hat{P}_{SE}(Y_f) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j} P(Y_f/X_{ij}) \tag{2.41}$$

where the second summation runs over all occurrences of the possibly final step in the course of jth simulation. An estimate of the variance may be calculated from

$$S^2 = \frac{N}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N}\hat{P}_i^2 - \hat{P}_{SE}(Y_f)\right] \tag{2.42}$$

where

$$\hat{P}_i = \sum_j P(Y_f/\vec{X}_{ij}) \tag{2.43}$$

The use of statistical estimation will always improve the variance of estimation, but it can be particularly useful if the probability of reaching the desired end point is small in all intermediate stages. It becomes essential when the probability of the end point becomes vanishingly small. If there were many intermediate stages which could, with very low probability, reach the desired end point, then statistical estimation might calculate the desired result with good accuracy.

Correlated Sampling

Concept of the Technique. Correlated sampling can be one of the most powerful VRTs due to the wide applicability of the technique, as well as the large efficiency gains which can be obtained. If the primary

objective of a simulation is to determine the effect of a small change in the system, crude Monte Carlo approach would make two independent runs, with and without the change in the system, then subtract the results obtained. Usually, the difference will be smaller than either of the two outputs, but the variance of the difference will be the sum of the variances in the two runs. In such cases, the use of correlated sampling can be essential to obtain statistically significant results. If the two simulations use a common random number at comparable stages in the computation, the correlation in results in the case of correlated sampling will reduce the variance of estimation much more. Another way of viewing correlated sampling through random number control is to realize that the use of the same random numbers will generate identical histories in those parts of the system which are the same, so that the difference in results will be due to the difference in the two systems. This will increase the efficiency compared to uncorrelated cases. Correlated sampling can be utilized in the following types of simulations:

* calculation of the effect of small change in the system

* difference in parameter in two or more similar cases is of more interest than its absolute values in them

* performance of parametric study of several similar problems

* the answers to unknown problems is to be estimated using the known answer of a similar problem.

Analytical Formulation of Correlated Sampling. Let the integrals I_1 , I_2 characterize two different but related problems.

$$I_1 = \int f(x)\ g_1(x)dx \tag{2.44}$$

and

$$I_2 = \int f_2(y)\ g_2(y)dy \tag{2.45}$$

If the main interest is the difference

$$\Delta = I_1 - I_2 \tag{2.46}$$

crude Monte Carlo approach will perform two separate simulations where, in the first, the estimator of I_1 is calculated as

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} g_1(X_i) \tag{2.47}$$

using a sample $X_1,\ldots,X_n$ selected randomly from $f_1(X)$,

and I_2 is estimated as

$$\hat{I}_2 = \frac{1}{N} \sum_{i=1}^{N} g_2(Y_i) \tag{2.48}$$

using a sample $Y_1,\ldots,Y_N$ selected randomly from $f_2(y)$. The difference is then estimated as

$$\hat{\Delta} = \hat{I}_1 - \hat{I}_2 \tag{2.49}$$

The variance in this case is

$$\sigma^2(\hat{\Delta}) = \sigma_1^2(\hat{I}_1) + \sigma^2(\hat{I}_2) - 2\,\mathrm{Cov}(\hat{I}_1,\hat{I}_2) \tag{2.50}$$

where

$$\sigma_1^2(\hat{I}_1) = E[(\hat{I}_1-I_1)^2] \tag{2.51}$$

$$\sigma_2^2(\hat{I}_2) = E[(\hat{I}_2-I_2)^2] \tag{2.52}$$

and

$$\mathrm{Cov}(\hat{I}_1,\hat{I}_2) = E[(I_1-I_1)(\hat{I}_2-I_2)]$$

$$= E[(I_1,I_2)] - I_1I_2 \tag{2.53}$$

Now if $\hat{I}_1$, $\hat{I}_2$ are positively correlated, then

$$\text{Cov}\,(\hat{I}_1,\hat{I}_2) \geq 0 \qquad (2.54)$$

and the variance in the correlated case will be less than that realized with no correlation.

History Reanalysis

Concept of the Technique. History reanalysis is essentially a form of correlated sampling, except that one does not actually run a second simulation using the same random numbers as in the first. Instead, the results of the first simulation are reanalyzed to calculate the answer for the second process (24). This technique reduces the variance due to correlation and cuts down the computational time involved since the second simulation is not actually performed. The technique can handle the same types of problems listed in the correlated sampling case with the condition that the differences in the systems being simulated must be expressible as a difference in probability distribution or in the scoring function.

Analytical Formulation. Assume that there are two problems which involve estimating I_1, I_2 as given by Eqs. (2.44) and (2.45). Assume also that a random sample $X_1,\ldots,X_N$ has been obtained from $f_1(X)$. The estimator for I_1 is as usual

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} g_1(X_i) \tag{2.55}$$

Since

$$I_2 = \int f_2(X)\, g_2(x)dx = \int f_1(x)\, \frac{g_2(x)f_2(x)}{f_1(x)}\, dx \tag{2.56}$$

an estimator for I_2 can be obtained using

$$I_2 = \frac{1}{N} \sum_{i=1}^{N} \frac{g_2(X_i)f_2(X_i)}{f_1(X_i)} \tag{2.57}$$

where $f_1(X_i) \neq 0$ whenever $g_2(X_i)f_2(X_i) \neq 0$. The sample variance of I_2 is

$$S_2^2 = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} [\frac{g_2(X_i)f_2(X_i)}{f_1(X_i)}]^2 - \hat{I}_2^2\} \tag{2.58}$$

To calculate the effect of correlation, it is necessary to estimate the variance of the difference directly. That is, if

$$\hat{\Delta}_i = \frac{g_2(X_i)f_2(X_i)}{f_1(X_i)} - g_1(X_i) \tag{2.59}$$

is the difference in the ith history and

$$\hat{\Delta} = \frac{1}{N} \sum_{i=1}^{N} \hat{\Delta}_i \tag{2.60}$$

is the average difference, then the sample variance is

$$S^2 = \frac{N}{N-1} \left[\frac{1}{N} \sum_{i=1}^{N} \hat{\Delta}_i^2 - \hat{\Delta}^2\right] \tag{2.61}$$

Control Variates

Concept of the Technique. In many simulation problems, there exist simplifications or approximations to the problem having analytic or closed form solutions. In these cases, the analytic information can be used to reduce variance by what is referred to as control variates. In this technique, the difference between the problem of interest and some analytical models approximating it is simulated. The gain in variance reduction or estimating accuracy is proportional to the degree of correlation between the true process and the analytical model used. This approach has a wide range of applicability and it is very useful when analytical representations of simplified models exist (24).

Analytical Formulation. Consider the integral

$$I = \int_{-\infty}^{\infty} g(x) f(x) dx \tag{2.62}$$

Assume that it is possible to get a function $h(x)$ whose expected value is known or can be analytically determined, and which approximates $g(x)$. If the value of the integral

$$\theta = \int_{-\infty}^{\infty} h(x)f(x)dx \tag{2.63}$$

is known, then the integral I can be expressed as

$$I = \int_{-\infty}^{+\infty} h(x)f(x)dx + \int_{-\infty}^{\infty} [g(x) - h(x)]f(x)dx$$

$$= \theta + \int_{-\infty}^{\infty} [g(x) - h(x)]\, f(x)dx = \theta + I_1 \tag{2.64}$$

The function $h(x)$ is called the control variate for $g(x)$. Since θ is known or can be calculated analytically, simulation is needed for estimation of I_1 ,

$$I_1 = \int_{-\infty}^{\infty} [g(x) - h(x)]\, f(x)dx \tag{2.65}$$

This can be performed using crude Monte Carlo by selecting a sample $X_1,\ldots,X_N$ from $f(x)$ and using

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} g(X_i) - \frac{1}{N} \sum_{i=1}^{N} h(X_i)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \hat{\Delta}_i \tag{2.66}$$

where

$$\hat{\Delta}_i = g(X_i) - h(X_i) \tag{2.67}$$

An estimate of the sample variance is

$$S^2 = \frac{N}{N-1}\left\{\frac{1}{N}\sum_{i=1}^{N}\hat{\Delta}_i^2 - \hat{\Delta}^2\right\} \tag{2.68}$$

where

$$\hat{\Delta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\Delta}_i \tag{2.69}$$

The efficiency of the control variates technique depends on the degree of similarity between $g(x)$, $h(x)$.

Antithetic Variates

Concept of the Technique. This technique is similar to the control variates approach, except that a negatively correlated function is chosen. This negative correlation is used to reduce the variance of estimation. Another difference between control variates and antithetic variates is that the expectation of the chosen function need not be known. Antithetic variates approach can be implemented in several methods. Two of these methods are discussed here.

Method I. If the parameter I where

$$I = \int_{\infty}^{\infty} g(x)f(x)dx \tag{2.70}$$

is to be estimated using antithetic approach, an unbiased estimator $\hat{I}_1$

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} g(X_i) \tag{2.71}$$

is found using crude sampling. A second unbiased estimator $\hat{I}_2$ for I is selected such that $\hat{I}_1$, $\hat{I}_2$ are negatively correlated. Linear combination of $\hat{I}_1$, $\hat{I}_2$ is a third unbiased estimator for I which can be, for example

$$\hat{\theta} = \alpha \hat{I}_1 + (1 - \alpha) \hat{I}_2 \tag{2.72}$$

In this method, α is chosen to be simply $\frac{1}{2}$, then

$$\hat{\theta} = \frac{1}{2} (\hat{I}_1 + \hat{I}_2) \tag{2.73}$$

will be an unbiased estimator for I with a variance given by

$$\sigma^2(\hat{\theta}) = \frac{1}{4} \sigma^2(\hat{I}_1) + \frac{1}{4} \sigma^2(\hat{I}_2) + \frac{1}{2} C_{ov}(\hat{I}_1, \hat{I}_2) \tag{2.74}$$

Since $\hat{I}_2$ is chosen to be negatively correlated with $\hat{I}_1$, then

$$Cov(\hat{I}_1, \hat{I}_2) \leq 0 \tag{2.75}$$

This could make the variance of the combined estimator $\hat{\theta}$ smaller than the variance of either of the two estimators $\hat{I}_1$, $\hat{I}_2$.

A convenient way to accomplish this method is to generate a set of random numbers $R_1, \ldots, R_N$ from $U(0,1)$ and two negatively correlated sets of random variables $X_1, \ldots, X_N$ and $X_1', \ldots, X_N'$ are obtained using the same set of random numbers where, for each selected random number R_i, the corresponding X_1', X_i' are calculated from

$$R_i = \int_{-\infty}^{X_i} f(x)\,dx \tag{2.76}$$

and

$$1 - R_i = \int_{-\infty}^{X_i'} f(x)\,dx \tag{2.77}$$

The negative correlation between each pair of values X_i, X_i' is clear; then the two estimators $\hat{I}_1$ and $\hat{I}_2$ will be negatively correlated. Defining

$$\theta_i = \frac{1}{2}\left[g(X_i) + g(X_i')\right] \tag{2.78}$$

the estimator of I using antithetic variates will be

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \theta_i = \frac{1}{2N} \sum_{i=1}^{N} [g(X_i) + g(\grave{X}_i)] \tag{2.79}$$

The sample variance is given by

$$S^2(\hat{\theta}) = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} \theta_i^2 - \hat{\theta}^2\} \tag{2.80}$$

Method II. In this method of implementing the antithetic approach, we try to find the value of α which best improves the estimation efficiency. This approach can be viewed as a combination of antithetic variates and stratified sampling where the range of $f(x)$ is divided into two strata, $-\infty < x < X_M$ and $X_M < x < \infty$. If a random number R_i is selected from $U(0,1)$, a pair of values X_i, $\grave{X}_i$ can be calculated from

$$\alpha R_i = \int_{-\infty}^{X_i} f(x)dx \tag{2.81}$$

and

$$\alpha+(1 - \alpha)R_i = \int_{-\infty}^{\grave{X}_i} f(x)dx \tag{2.82}$$

which means that X_i is selected from the range $-\infty < x < X_M$ and $\grave{X}_i$ from $X_M < x < \infty$. Also, X_i, $\grave{X}_i$ are negatively correlated. In this case, the combined variable will be

$$\theta_i = \alpha\, g(X_i) + (1 - \alpha)\, g(\hat{X}_i) \tag{2.83}$$

and the new unbiased estimator for I is

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \theta_i = \frac{1}{N} \sum_{i=1}^{N} [\alpha g(X_i) + (1-\alpha) g(\hat{X}_i)] \tag{2.84}$$

with the sample variance

$$S^2 = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} \theta_i^2 - \hat{\theta}^2\} \tag{2.85}$$

If α is properly selected, this method can give much better results than the first simple one. Selection of α is a difficult task, but a rule of thumb is to select α such that

$$g(X_M) = \alpha\, g(X_L) + (1-\alpha) g(X_U) \tag{2.86}$$

where X_U and X_L are the upper and lower limits of the range $f(x)$. Alternatively, α can be determined using a trial and error method to obtain the optimum efficiency.

Regression. Regression techniques can be applied to a wide variety of Monte Carlo simulations to produce unbiased estimators for a set of parameters (integrals) when correlation between them is known to exist. Regression

technique will make use of this correlation to reduce the variance of estimation.

Analytical Formulation. If a set of integrals $I_1,\ldots,I_p$ are to be estimated, regression can be applied to determine the minimum variance unbiased estimators by a set of estimates $\hat{\theta}_1,\ldots,\hat{\theta}_n$ $(n \geq P)$ such that

$$E(\hat{\theta}_j) = a_{ji} I_1 + \ldots + a_{jp} I_p \quad , j=1,\ldots,n \quad (2.87)$$

where a_{ji} , $j=1,\ldots,n$ and $i=1,\ldots,P$ is a set of known constant. If the coefficients a_{ji} in Eq (2.87) is represented in matrix form,

$$\vec{A} = \begin{bmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1P} \\ a_{21} & \cdot & \cdot & \cdot & a_{2P} \\ a_{n1} & \cdot & \cdot & \cdot & a_{nP} \end{bmatrix} \quad (2.88)$$

and a sample consisting of N independent sets of simulated values for θ_j , then one calculates

$$\hat{\theta}_j = \frac{1}{N} \sum_{i=1}^{N} \theta_{ij} \quad , \quad i=1,\ldots,n \quad (2.89)$$

to construct the column matrix

$$\hat{\vec{\theta}} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \cdot \\ \cdot \\ \hat{\theta}_n \end{bmatrix} \tag{2.90}$$

Now, an estimate for the set of integral I where

$$\vec{I} = \begin{bmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ I_P \end{bmatrix} \tag{2.91}$$

is given by $\hat{\vec{I}}$,

$$\hat{\vec{I}} = (\vec{A}^T \vec{V}^{-1} \vec{A})^{-1} \vec{A}^T \vec{V}^{-1} \hat{\vec{\theta}} \tag{2.92}$$

where

$$\vec{V} = \begin{bmatrix} V_{11} & \cdot & \cdot & \cdot & V_{1n} \\ V_{21} & \cdot & \cdot & \cdot & V_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ V_{n1} & \cdot & \cdot & \cdot & V_{nn} \end{bmatrix} \tag{2.93}$$

is the covariance matrix for $\hat{\theta}_1,\ldots,\hat{\theta}_n$ and A^T is the transposition of A . That is,

$$V_{ij} = E\left[\{\hat{\theta}_i - E(\hat{\theta}_i)\}\{\hat{\theta}_j - E(\hat{\theta}_j)\}\right] \tag{2.94}$$

$$i=1,\ldots,n \quad , \quad j=1,\ldots,n$$

which could be estimated as

$$\hat{V}_{ij} = \sum_{k=1}^{N} (\theta_{ki} - \hat{\theta}_i)(\theta_{kj} - \hat{\theta}_j) \tag{2.95}$$

$$i=1,\ldots,n \quad , \quad j=1,\ldots,n$$

where $\hat{\theta}_i$ is calculated from Eq (2.90), and

$$\hat{\vec{V}} = \begin{bmatrix} V_{11} & \cdot & \cdot & \cdot & V_{1n} \\ V_{21} & \cdot & \cdot & \cdot & V_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ V_{N1} & \cdot & \cdot & \cdot & V_{nn} \end{bmatrix} \tag{2.96}$$

The new unbiased estimator is

$$\hat{I}^* = (\vec{A}^T \hat{\vec{V}}^{-1} \vec{A})^{-1} \vec{A}^T \hat{\vec{V}} \hat{\theta} \tag{2.97}$$

Generally, it is difficult to formulate the estimators $\hat{\theta}_1,\ldots,\hat{\theta}_n$. This limits the applicability of the method in real situations.

Specialized Techniques

This group of VRTs includes those techniques which are useful to a specific kins of problem. Some of those techniques are not well developed or extremely specialized. However, some of those techniques are the only way to get a considerable variance reduction in certain cases. In this section, the most common techniques of the above characteristics will be discussed.

Conditional Monte Carlo. In some cases, is is hard to estimate the parameter

$$\theta = E[\phi(\alpha)] \tag{2.98}$$

where α is a random vector distributed over a space A with a probability density function $f(a)$. This may be due to the complexity of $f(\alpha)$. One way to deal with such a problem is to embed the space A in a product space $C = A \times B$ where B is suitably chosen. Each point in C can be expressed as

$$c = (a,b) \tag{2.99}$$

and a can be considered as a function of c which maps the points of C to A. If we sample a random vector $\gamma = (\alpha,\beta)$ from C with a probability density function $h(c)$, a mapping of γ to α is obtained which is a random vector of A. In general, α will not have the desired density function f, so an appropriate weighting function should be used to compensate for that.

If we choose $g(c) = g(a,b)$, an arbitrary real function defined on C such that

$$G(a) = \int_{\beta} g(a,b)\, db \neq 0 \quad \text{for any } (a,b) \tag{2.100}$$

and

$$h(c) \neq 0 \quad \text{for any } (c) \tag{2.101}$$

a suitable weighting function could be

$$w(c) = f(a)g(c)J(c) \;/\; G(a)h(c) \tag{2.102}$$

where $J(c)$ is the Jacobian of the transformation $c=(a,b)$ which can be written as

$$J(c) = J(a,b) = \frac{dadb}{dc} \tag{2.103}$$

Since a is the first coordinate of c, the following identity holds (11)

$$\int_A \phi(a)\, f(a)da = \int_A da\, \frac{\phi(a)f(a)}{G(a)} \int_B g(a,b)db$$

$$= \int_{AxB} \frac{\phi(a)f(a)g(c)}{G(a)h(c)}\, h(c)dadb$$

$$= \int_C \phi(a)w(c)h(c)\, \frac{dadb}{J(c)}$$

$$= \int_C \phi(a)w(c)b(c)d(c) \tag{2.104}$$

This shows that, if a is the first coordinate of a random vector γ sampled from C with density function $h(c)$, then

$$t = \phi(\alpha)\, W(\gamma) \tag{2.105}$$

is an unbiased estimator of θ. Here B, h are chosen arbitrarily to simplify the sampling procedure. The function g acts as an importance function which should be selected to minimize the variation in t, and hence increase the precision of estimation.

Conditional Monte Carlo is a special case of the above theory where $h(c)$ is a given distribution on $C = A \times B$ and $f(a) = f(a,b_o)$ is the conditional distribution of $h(c)$ given that $b = b_o$ we have

$$h(c)d(c) = f(a,b)\, \psi(b)\, dadb \tag{2.106}$$

where $\psi(b)$ is the probability density function of B and $\gamma = (\alpha,\beta)$ has the density function $h(c)$. In this case

$$J(c) = h(c) \,/\, f(a,b)\, \psi(b) \tag{2.107}$$

and for a given b_o

$$J(a,b_o) = h(a,b_o) \,/\, f(a)\ \psi(b_o) \tag{2.108}$$

Eliminating $f(a)$ from the weight function, we get

$$w(c) = \frac{h(a,b_o)J(a,b)g(a,b)}{h(a,b)J(a,b_o)\psi(b_o)G(a)} \tag{2.109}$$

This leads to the following rule. Let $\gamma = (\alpha,\beta)$ be distributed over C with probability density function

$$h(c) = h(a,b) \tag{2.110}$$

Then

$$t = \phi(\alpha)w(\gamma) \tag{2.11}$$

where $w(\gamma)$ is given by Eq (2.111) as an unbiased estimator of the conditional expression of $\phi(\alpha)$ given that $B = b_o$. It is clear that this approach requires neither sampling from space A which may be awkward, nor evaluation of the possibly complicated function f. Besides, we can achieve variance reduction in the estimation if g is suitably selected.

Sequential Sampling. This is not a specific variance technique, but rather a general approach to the use of other techniques. It is useful to apply this technique when there is little or no apriori information about the expected results of the simulation. In this technique, a series of sequential simulation runs is performed to reduce the variance of estimated parameters. In the first run, little or no reduction variance is achieved. In the second run, parameters estimated in the first one are used for applying other VRTs such as importance sampling, Russina Roulette, splitting, or stratified sampling. A third run can then be made using the improved sampling parameters and this "self-learning" process can be carried out repeatedly with the efficiency of sampling improving at each stage. In spite of the simplicity of this approach, little work on sequential sampling has been done (24). Considering this technique a trade-off must be done between the required extensive computation and the efficiency gain from improved sampling. The sequential nature of this technqiue may lead to more underbiased or overbiased estimation, if the initial choice of the parameter is biased.

Orthonormal Functions. This VRT can be very useful when applied to multidimensional problems. In this method, a set of orthogonal functions over a

region of multiple integration is defined, then a sampling scheme is structured to permit efficient sampling over this region from a joint probability density function. The procedures to implement this technique are not well developed, but the potential gain when applying it is still high.

Adjoint Method. Frequently, when building a simulation model, one can find a set of mathematical equations which is adjoint or inverted with respect to the original set. In such cases, a solution for one set of equations will give the solution for the second set. The basic idea in applying the adjoint method as a VRT is to simulate the adjoint set of equations which does not represent any real process, but is easier or more efficient to simulate. It would give a solution which helps in estimating the original parameter direclty or in applying another VRT. In some cases, one can divide the problem into two parts; in one of them, the adjoint method is applied while direct simulation is applied in the second part.

The adjoint method has been exploited very successfully in radiation transport problems because of the precise formulation of this problem as a linear integral equation for which an adjoint formulation can be obtained (24).

This technique needs more investigation and further development to be generally applicable in simulation.

Transformations. This method is a special form of importance sampling which is characterized by formulating the priori information about the process in a parametric, closed form representation. That information can be used to alter the sampling procedure by transformation. This method has been largely employed in radiation transport calculations where the function of interest have an approximately exponential form (24).

III. Selection of Variance Reduction Techniques

In the preceding chapter, basic concepts of different VRTs were discussed. Those techniques are not equally efficient when applied to a specific problem. The selection of a promising technique for a particular problem can cause considerable difficulty due to the large number of possibilities available. This chapter is devoted to set general rules for selection of the appropriate technique(s) for a certain situation or, in other words, to show where each technique can generally be used. To achieve this goal, a summary of properties and concepts of most of the known VRTs is given in a tabular form to help in selecting the suitable technique(s).

For the analyst to select and implement an appropriate variance reduction technique or techniques, the following systematic procedure should be applied:

1. Definition of the problem information that can be used as a basis to select an appropriate technique(s).
2. Selection of specific technique(s) that should be considered for a given problem.
3. Setting of basic guidelines to implement the selected procedure.

These aspects are described in the following three sections.

Definition of Problem Information

The efficiency of a variance reduction technique is strongly related to the efficiency of the use of the known information about the problem. Primarily, it is essential to characterize the aspects of the problem that can help to indicate the fruitful variance reduction technique for this problem. Helpful information items are organized in the following table. These items are the basic items needed for most of the techniques.

Table 3.1 presents the required useful information that should be known prior to the selection and implementation of the suitable variance reduction technique.

Selection of Variance Reduction Technique(s)

The most difficult step in utilization of variance reduction techniques is the selection of the suitable technique(s) that would fit the problem of interest and give an effective variance reduction or reduce the required sample size for a given degree of accuracy. This difficulty can be reduced by preparing and utilizing the information about the problem listed in Table 3.1 and understanding the characteristics of different available variance reduction techniques. For that reason, a comprehensive summary of variance reduction techniques is presented in this section. Having the information about the problem under investigation in mind, and understanding

TABLE 3.1

PROBLEM INFORMATION

NEEDED FOR SELECTION OF SUITABLE TECHNIQUE

1. Definition of nature of the problem relative to:
   - expected values (means, variances, etc.) to be estimated
   - sensitivities or variations of parameters of interest
   - possible mathematical formulations (integral equations, expected values, etc.)
   - any sequential characteristics, such as independent path; outcomes depend on intermediate step
   - input conditions which are random variables to be sampled.
2. Identification of portions of problem or parameter to be estimated that can be:
   - expressed in an analytical form such as single integral, multiple integral, differential and/or integral equations
   - solved analytically, such as expected values, variances, probabilities, etc.
   - represented by approximate, simplified positively correlated analytical expressions
   - represented by approximate, simplified negatively correlated analytical expressions
   - established as relatively not important to final outcomes compared to other aspects of the problem.
3. Identification of variables in the problem which:
   - are very important to the expected outcome
   - are not expected to significantly impact the results
   - are strongly correlated with other variables.
4. Location of final events or outcomes of the problem which:
   - have very small probabilities
   - have very large probabilities
   - have outcomes relatively insensitive to problem parameters
   - have known probabilities of occurrence from intermediate stages in the problem
   - are linear combinations of other events or random variables
   - have known correlation with other events or outcomes.

the basic idea of each available variance reduction technique, one can use the following summary of the characteristics of different techniques to select the techniques that could fit his problem. With the revision of the problem information and selected technique(s) characteristics, the most suitable technique(s) could be identified.

The following summary of variance reduction techniques' characteristics includes description, supposed criteria of application, advantages and disadvantages of each technique (24). It also includes the typical area of application of each technique. It should be noted that, in many cases, more than one technique can be separately applied to the problem, but each of them will deal with the problem in different approaches. Also, in some cases more than one technique may be applied to solve one problem in the same time. Each of these techniques will be used in one stage of the problem.

The most important point to keep in mind when selecting and implementing one or more of the variance reduction techniques for a certain problem is that the applied technique(s) will reduce the variance of only one parameter or aspect of the problem being simulated. Using variance reduction techniques designed for one parameter will usually reduce the effectiveness of the simulation to estimate other parameters. Therefore, it is very important

to determine all of the results which will be desired from the simulation before searching for a suitable variance reduction technique. When more than one quantity is to be estimated, the chosen technique(s) should not degrade the efficiency of any of the estimations. In many situations, it may be advisable to implement a different variance reduction technique for each parameter.

TABLE 3.2

Characteristics of Variance Reduction Techniques

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Importance Sampling	Sampling procedure is modified such that the more important events are sampled more frequently	Certain events are known to be important Estimated parameters are based on events with very low probability of occurrence Certain events are not of interest Regions of importance can be defined	It can give great improvement with limited effort Well developed procedures Easy implementation	Can give worthless results if not properly applied A priori knowledge is needed Additional computer time is needed to implement the technique	Most commonly used method. Application in PERT, reliability fault tree analysis, queuing and radiation transport Applicable to almost any problem having criteria indicated.
Russian Roulette and Splitting (RR&S)	Use of probabilities to kill samples in an uninteresting portion of the simulation (RR) and to increase number of samples in regions of interest (S)	Sequential process Events with low probabilities are involved Relative importance of all outcomes is known Useful where importance sampling or expected values are used	Very limited computation Convenient way to accomplish crude importance sampling	Requires a priori knowledge of importance of intermediate stages May not be effective if used alone Splitting may be difficult to implement	Particle transport, search problems, network analysis, traffic flow and random walk. Usually when low probability events or unimportant events are involved.

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Systematic Sampling	Selected random numbers are systematically distributed over the sample space	Random variables are to be generated at the beginning of the problem (Source or Initial Conditions) Relative importance of ranges of sampled variables is unknown	East to implement Little additional computation No risk in application Always gives variance reduction	Often gives marginal improvement Effective only in limited number of situations	Could be useful in any problem where random inputs are of interest Examples are in queuing and reliability
Stratified Sampling	Sample space is divided into sections. Sampling is accomplished from each section. More important areas are emphasized	Random variables are to be generated at the beginning of the problem Certain ranges of the variables are more important than the other	Easy to implement Can give great improvement Gives better results than systematic if properly applied Can be optimally applied	Requires a priori knowledge of importance Can lead to worse results if not properly applied Sometimes difficult to define sampling ranges	Queuing Any problem with random conditions such as reliability and network analysis

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Expected Value	Include known expected values in the process to replace a stochastic portion of a simulation	Analytical results, probabilities, or expected values of the problem are known Higher moments of the above mentioned portions are not essential to overall simulation	Almost always gives improvement Easy to implement	May require prohibitive analysis or computation	PERT, Radiation transprot Limited in other areas Could be used where known results are available for parts of the problem
Statistical Estimation	Include known expected values or probabilities in estimated and not in simulation model	Analytical results, probabilities or expected values of portions of the problem are known Stochastic nature is essential to overall simulation Process is sequential in nature Probability of final outcome is known but small at all intermediate stages	Always gives improvement Can get great improvement Could be the only way to get an answer	May require prohibitive analysis or computation	Primary application in radiation transport Potential to use at any simulation

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Correlated Sampling	Some of random numbers are used in successive trials to provide correlation among the outcome in successive trials	Sensitivities studies are of interest Effect of parameter variations to be determined Small effects are of interest	Easy to comprehend Applicable to many situations Little risk when applied Often the only way to detect small effects	May be difficult to implement Sensitive to specific applications	Can be used in any problem where sensitivities are of interest Examples are network design, tactics evaluation, reliability and queuing
History Reanalysis	Takes results of one simulation and reinterprets as results of second simulation weighting answers to remove bias	Results of two or more similar problems are needed Difference in problems may be written as difference in probability distributions	Results are highly correlated thus reducing variance in difference Save computation time for second simulation	Problems may differ in aspects more than probability distribution Variance in the seocnd problem may be prohibitively high in history reanalysis	Potentially many applications. It was used in radiation transport

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Control Variates	Using simplified or approximate positively correlated representation for undetermined parts of simulation	Portions of the problem can be approximated with an analytical representation A simple low variance approximate simulation is known	Requires simple calculation Provides considerable flexibility Can give great improvement	Requires understanding of the process Simple analytic approximation may not exist	Potentially can be used at any area where approximate representation of the random variable under estimation is known Has been applied in queuing problems
Antithetic Variates	Using simplified or approximate negatively correlated representation for undetermined parts of the problem	Analytical or approximate representation are known and negatively correlated with the problem variables	Requires simple calculation Provides considerable flexibility Can give great improvement	Requires understanding of the process Costs more then control variates	Could be used in reliability, queuing, sensitivity analysis Has been applied to queuing problems

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Regression	Used with linear combinations of parameter being estimated in a simulation model. An approximate minimum variance estimate is included exploiting the inherent correlation among the parameters	Estimated parameters are linear combinations of random variables Correlation among variables is known to exist	Little or no bias results Exploits correlation Little loss if there is no correlation Requires little information about the problem	Increased computation Limited to linear combination of the random variables in its usual form	Almost any problem where the outcome is a linear combination of statistical estimators (Reliability, queuing)
Sequential Sampling	Developing of estimate for parameter to be used in other VRTs by using information generated from previous sampling	Little or no information is available about the process Other variance reduction techniques are to be implemented	Can be almost always applied Requires little or no prior information Can be used in conjunction with other techniques	May require considerable calculations Efficiency may be very low May not be useful in certain problem (with low probabilities) Not well developed	Evaluation of multiple integrals, estimation of distribution parameters and could be used in lifetime studies

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Orthogonal Functions	The orthogonal characteristics of certain orthonormal functions are used to formulate a sampling scheme with reduced variance	Multidimensional variables in the problem Multiple integral formulations can be determined	Great possibilities for problems with multidimensional integral	Not well developed for general applications Difficult to implement efficiently Requires considerable calculations Involves sampling from joint random variables	Multiple integrals Applied in nuclear physics Applied when multiple integrals are possible in complex systems
Adjoint Method	The problem is simulated in a backward or reverse sense (mathematical adjoint formulation)	Reverse process can be formulated Starting positions for final outcomes can be defined Mathematical adjoint formulation is known	Can give indication of what importance sampling is required Very powerful when it can be applied Useful when studying system input sensitivity	Not well developed for all applications Difficult to implement Adjoint concept is difficult to formulate for most problems	Radiation transport Could be used in queuing, inventory, etc. if problem can be formulated as integral or differential equations

TABLE 3.2 (Cont'd)

VRT	DESCRIPTION OF TECHNIQUE	CRITERIA OF APPLICATION	ADVANTAGES	DISADVANTAGES	TYPICAL APPLICATIONS
Transformation	Use a transform of the process being simulated to give a problem with reduced variance when simulated	Approximations are known for parts of the problem An analytical formulation is known Transformed equation can be easily simulated	Permits parametric representation of knowledge about the system Can lead to substantial improvements	Not well developed for most applications May require complete analytical statement of the problem Appears to be limited in scope of application	Used in radiation transport problems Can be used in queuing, inventory, etc.
Conditional Monte Carlo	Imbeds problem in a larger process using conditional probabilities in larger process	Process is complicated, but can be embedded in a simpler simulation	When applicable, can give great improvement Some algorithms are available for network application	Sometimes difficult to recognize applicability	Stochastic PERT, CPM, and inventory

IV. Implementation of Variance Reduction Techniques

Once the suitable VRT is selected to be applied for a specific simulation problem, a plan for implementing the selected technique should be made. In many cases, the theory given in the literature cannot be directly used to implement the selected technique. In this chapter, general guidelines to implement the more important VRTs are given in the form of a step-by-step procedure. Simple examples, to show how to apply these steps, are presented for the most commonly used techniques. Computer programs were written in FORTRAN V to accomplish those examples. These examples would give better insight of the implementation and the efficiency of the techniques.

Importance Sampling

Guidelines for Implementation. The general guidelines that could be followed to implement importance sampling are as follow:

1. Express, if possible, the expected value being estimated as

$$I = \int g(x) f(x) dx \tag{4.1}$$

where x is the random variable of importance sampling and $f(x)$ is its density function.

2. Determine the functional form of $g(x)$ analytically or by selecting values of x and estimating the corresponding $g(x)$.
3. Select an importance function $f^*(x)$ which approximates $f(x)g(x)$.
4. Select a random sample $X_1,\ldots,X_N$ from $f^*(x)$ using a set of random numbers from $U(0,1)$.
5. Estimate I using (2.5).
6. The estimator for the sample variance can be calculated using (2.6).

Example. If it is required to estimate the probability $P(x \leq 1)$ when $f(x)$ is given by

$$f(x) = .01 \exp(-.01x) \tag{4.2}$$

the crude Monte Carlo will require hundreds of sample values from $f(x)$ to accurately determine $P(1)$. This is due to the fact that x will be less than 1 in approximately 1/100 of the sample values. Applying importance sampling method could reduce the required sample size for a given degree of accuracy. Following the above guidelines to implement importance sampling, the value of $P(1)$ can be estimated as follows:

1. Express the required integral as

$$I = \int_0^1 f(x)dx \tag{4.3}$$

where $f(x)$ is given by (4.2).

2. Let $f^*(x)$ be

$$f^*(x) = \exp(-x) \tag{4.4}$$

The selected importance function $f^*(x)$ will give the opportunity for x to take values less than or equal to 1 more frequently.

3. Select a random number R_i from $U(0,1)$, then determine the corresponding X_i using

$$X_i = F^{*-1}(R_i) = \log_e(1/(1-R_i)) \tag{4.5}$$

4. Determine the values of $f(X_i)$, $f^*(X_i)$.
5. Repeat steps 3 and 4 for N times where N is the chosen sample size.
6. Determine the estimator for $P(1)$ as

$$P(1) = \sum_{i=1}^{N} \frac{f(X_i)}{f^*(X_i)}, \quad x_i \leq 1 \tag{4.6}$$

7. Determine the sample variance using (2.6).

The above steps were coded in a computer program using

FORTRAN V. The code and the results are given in Appendix A. The results are given for $N = 20$. This small sample size gave an accurate estimation of $P(1)$ as .01015 (the theoretical value is .00995). The sample variance was .0252 . If crude Monte Carlo were used, it would require hundreds of sample values to give the same accuracy.

Russian Roulette and Splitting

Guidelines for Implementation. To apply Russian Roulette and Splitting, the following general steps can be followed:

1. Determine stages of the problem in which possible conditions can be divided into N regions where each of them contains points of roughly the same importance.
2. For each region choose average weight standards, W_{A_i} , $i=1$, N . This weight should be inversely proportional to the importance of the region.
3. If no other VRTs are used, set high and low weight standards, W_{H_i} , W_{L_i} , equal to W_{A_i} ; otherwise, W_{H_i} , W_{L_i} should be sufficiently spaced above and below W_{A_i} . The spacing should prevent any unnecessary Russian Roulette or Splitting and assign approximately equal weights to histories of roughly the same importance.

4. When a history arrives at a particular stage in region R_i with weight W, carry out the suitable action according to the following cases:

   a. If $W < W_{L_i}$, apply Russian Roulette by killing the history with probability $1 - \frac{W}{W_{A_i}}$ or letting it continue in simulation with probability W/W_{A_i} carrying a new weight W_{A_i}.

   b. If $W_{L_i} < W < W_{H_i}$, let the history continue with weight W.

   c. If $W > W_{h_i}$, carry out splitting as follows:

      (i) Determine n such that

      $$0 \leq W - n\, W_{A_i} < W_{A_i}$$

      (ii) Split the history into n "daughter" histories, starting in that point with weight W_{A_i}.

      (iii) With probability $\frac{W - n\, W_{A_i}}{W_{A_i}}$, create one more daughter history with weight W_{A_i}.

5. Form estimate $\hat{I}_i$ for each history i

$$\hat{I}_i = \sum_{\rho,\ \text{daughter of } i} g(\vec{x}_\rho)\, W_\rho \qquad (4.7)$$

6. The final estimate of I is given by

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} \hat{I}_i \tag{4.8}$$

where N is the number of starting histories.

7. The variance of the sample is given by (2.20).

Example of Using Russian Roulette in System Reliability. Rice and Moore introduced a Monte carlo technique for estimating lower confidence limits on system reliability (26). The algorithm which they gave to accomplish crude Monte Carlo requires a generation of total 3000 random variates from the normal distribution $N(0,1)$. Russian Roulette can be applied to reduce the number of sample values generated from $N(0,1)$ when using this algorithm.

Case 1: Series Connection of Subsystems. Consider a system of three subsystems connected in series. The reliability of each subsystem P_i is calculated using pass or fail test as

$$P_i = \frac{1 - F_i}{N} \tag{4.9}$$

where P_i is the number of failures in N trials. One can roughly say that

$$r_s = P_1 P_2 P_3 \tag{4.10}$$

where r_s is the system reliability. In fact, each P_i is a random variable which can be considered asymptotically normal (26), accordingly R_s will be a random variable depending on P_1, P_2 and P_3. If it is required to determine $P(r_s \leq R_x)$ or $P(r_s \geq R_x)$, a modified algorithm using Russian Roulette can be carried out as follows:

1. Using the data from pass-fail tests, the first estimate of reliability of each subsystem can be calculated as

$$\hat{P}_i = 1 - F_i/n_i \tag{4.11}$$

the asymptotic variance is

$$V_i = \frac{\hat{P}_i \hat{q}_i}{n_i} \tag{4.12}$$

where $\hat{q}_i = 1 - \hat{P}_i$, n_i being the number of pass-fail tests of the subsystem i.

2. Draw a random variable X_i from $N(0,1)$.
3. Calculate the second estimate P_i as

$$\hat{\hat{P}}_i = \hat{P}_i + V_i \times X_i \quad , \quad i=1 \tag{4.13}$$

4. If $P_1 \leq R_x$, skip steps 5 through 8.
5. Calculate $\hat{\hat{P}}_2$ as in steps 2 and 3 where i=2
6. Calculate R_{12}

$$r_{12} = \hat{\hat{P}}_1 \hat{\hat{P}}_2 \tag{4.14}$$

If $r_{12} \leq R_x$, skip steps 7 and 8.

7. Calculate $\hat{\hat{P}}_3$ as in steps 2 and 3, where i=3 .
8. Calculate r_s as

$$r_s = \hat{\hat{P}}_1 \cdot \hat{\hat{P}}_2 \cdot \hat{\hat{P}}_3 \tag{4.15}$$

If $r_s \leq R_x$, then

9. Add 1 to the number of trials where $r_s \leq R_x$.
10. Start a new trial at step 2 and repeat for 1000 trials.

Notice that the application of Russian Roulette reduces the number of generated normal random variates to one in some trials and two in other trials. In some trials, we have to generate three random variates from normal as in the original algorithm. The total number of reductions depends on the values of $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$ and R_x .

The above algorithms have been coded to a computer program in FORTRAN V which is attached, along with the results, in Appendix B. The attached results are obtained

when $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_3$ are .7, .8 and .8, respectively. and R_x runs from .3 to .75. The number of reductions increases with the increase of R_x . When R_x is .75, a total of 1654 out of 3000 generations from normal N(0,1) were saved.

Case 2: Parallel Connection of Subsystems. The above algorithm can be used to handle systems consisting of parallel subsystems with some modifications. Suppose that we have a system with three parallel subsystems which were examined using pass-fail tests. If it is required to calculate $P(r_s \geq R_x)$, one can use the same algorithm given for the series case to apply Russian Roulette to simulation with the following modifications:

a. In step 4, we skip to 9 if $\hat{\hat{P}}_1 \geq R_x$

b. In step 6, r_{12} is calculated as

$$r_{12} = 1 - (1 - \hat{\hat{P}}_1)(1 - \hat{\hat{P}}_2) \tag{4.16}$$

and we skip to step 9 if $r_{12} \geq R_x$

c. In step 8, r_s will be calculated as

$$r_s = 1 - (1 = \hat{\hat{P}}_1)(1 - \hat{\hat{P}}_2)(1 - \hat{\hat{P}}_3) \tag{4.17}$$

and we check if $r_s \geq R_x$

d. In step 9, we add 1 to the number of trials where $r_s \geq R_x$.

The computer program for the parallel case is presented in Appendix C, along with the results for $\hat{P}_i$ equal to .7, i=1,2,3 and R_x runs from .905 to .950 . The results showed that the number of reductions decreases when R_x increases. When R_x was .905, a total of 573 reductions in the number of generated random variates from N(0,1) out of a total 3000. In another experiment, more reduction was achieved as R_x was decreased.

Systematic Sampling

Systematic sampling can be implemented using the following steps:

1. Determine the cumulative function for f(x) . Divide its range into N intervals, each of width 1/N . N should be between 5 and 50.
2. Generate n sets of N random numbers, each from U(0,1) . Denote them by $R_{11},\ldots,R_{1N}$; $R_{21},\ldots,R_{2N}$; $R_{n1},\ldots,R_{nN}$.
3. Allocate the generated random numbers into the corresponding intervals using

$$R_{ij}^{\wedge} = \frac{j - R_{ij}}{N} \quad \begin{matrix} i=1,\ldots,n \\ j=1,\ldots,N \end{matrix} \qquad (4.18)$$

4. Determine X_{ij} corresponding to each R_{ij} from

$$R_{ij} = \int_{-\infty}^{X_{ij}} f(x)\,dx \tag{4.19}$$

5. Estimate the integral I using (2.23) and (2.24).

6. Estimate the sample variance using (2.25).

Example. Suppose it is required to estimate the value of the integral

$$I = \int_0^1 \frac{e^x - 1}{e - 1}\,dx \tag{4.20}$$

using Monte Carlo techniques. In fact, the value of this integral can be easily calculated, but it will be used to demonstrate how to implement several VRTs. Let the integrand of (4.20) be

$$f(x) = \frac{e^x - 1}{e - 1} \tag{4.21}$$

The crude Monte Carlo procedure to estimate I requires the generation of N random numbers from U(0,1) , then determination of the values of f(x) at these points. That is because the range of integration runs between 0 and 1. If the number of generated random numbers N is small, crude Monte Carlo will give an inaccurate estimate of I with a large variance of sample. However, one can

improve the accuracy of estimation and reduce the sample variance by using one of the VRTs. Systematic sampling can be applied to estimate I as follows:

1. Divide the range of integration (0-1) to N intervals each of width .25
2. Generate n random numbers from U(0,1) for each interval, then allocate them inside that interval using (4.18). Determine the value of f(x) at this point.
3. The estimator of I is calculated using (2.23) and (2.24).
4. The sample variance S^2 is calculated using (2.25).

A computer program in FORTRAN V was written to estimate the integral I given by (4.20) using crude Monte Carlo, systematic sampling and stratified sampling. The purpose of that is to demonstrate the implementation of the three techniques and to compare the results when the same sample size and the same random number stream are used. The results for the three methods are attached, along with the program code, at Appendix D. These results will be discussed in the next section.

Stratified Sampling

Stratified sampling can be implemented using the following steps:

1. Divide the range of the variable being simulated into N intervals of length $L_1,\ldots,L_N$. N should be between 5 and 50. L_j can be selected so that the variation of integrand is approximately the same inside the interval j .

2. Determine P_j , the probability that x will be in L_i as

$$P_j = \int_{x\varepsilon L_j} f(x)dx \ , \ j=1,\ldots,N \qquad (4.22)$$

3. Determine the number of sample values, n_j , $j=1$, N taken from each interval using (2.34) and (2.35).

4. For each interval j , select a set of n_j random numbers R_{ij} , $i=1$, n_j from $U(0,1)$. Allocate these random numbers in the specific interval j and calculate the corresponding values of X_{ij} using

$$R_{ij}P_j + \sum_{\rho=1}^{j-1} P_\rho = \int_{-\infty}^{X_{ij}} f(x)dx \qquad (4.23)$$

5. Estimate I using (2.30) and (2.31).

6. Calculate S^2 using (2.32).

Example. It is required to estimate the value of the integral given by (4.20) using stratified sampling. The

above steps can be used to implement the technique as follows.

1. The range of integration (0-1) was divided into N = 4 intervals.
2. The length of each interval L_i was chosen to have the same variation of f(x) in each interval. The intervals turn out to be (0-.36) , (.36-.62) , (.62-.83) , (.83-1) .
3. For each interval, four random numbers were generated from U(0,1) and allocated inside the interval. The value of f(x) corresponding to each random number was calculated.
4. The estimator of I inside each interval j was calculated using (2.31).
5. The final estimator of I was calculated using (2.30).
6. The variance of estimation was calculated using (2.32).

The computer program in Appendix D performs these steps to implement stratified sampling. However, it also performs systematic sampling and crude Monte Carlo as stated before. Results for the three methods are also given in Appendix D. From these results, it is clear that the stratified sampling method gives the best estimation among the three methods. The estimator of I (= .4153) is the

nearest to the exact value of I $(= .418)$. Also, the method gives the lowest sample variance $(.0174)$. In fact, the estimations of I given by crude Monte Carlo, systematic sampling and stratified sampling were .3834, .4396, .4153 and the sample variances were .1198, .0220, .0174, respectively. These results give an idea about the efficiency of each of the three methods, taking into consideration the additional computational effort done in the last two methods.

Expected Value

When expected value technique is applicable to a certain simulation problem, the procedure to implement the technique will differ according to the role of the random process, which can be replaced by its expected value, in the overall simulation. In fact, one of the following cases will fit the problem under consideration. For each possible case, guidelines to use the technique are given.

1. If the process to be replaced by its expected value is a selection of a random variable y from a density function $f(y)$, set

$$y = E[f(y)] \tag{4.24}$$

and continue the simulation.

2. If the process represents a decision between terminating or not terminating the history, let

the history continue in simulation carrying a weight W_{new} given by

$$W_{new} = W_{old} \cdot P_s \tag{4.25}$$

where P_s is the probability of survival of the history at the decision point and W_{old} and W_{new} are the weights of the history before and after the replaced random process. For any parameter being calculated, an estimate for each history can be made by summing the contributions from that history; that is,

$$\hat{I}_i = \sum_j W_{ij}\, g(x_{ij}) \tag{4.26}$$

where W_{ij} is the weight of the ith history at the time of the jth contribution to the final result. The final estimate for the parameter is given by

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} I_i \tag{4.27}$$

and the sample variance is given by

$$S^2 = \frac{N}{N-1} \left\{ \frac{1}{N} \sum_{i=1}^{N} I_i^2 - \hat{I}^2 \right\} \tag{4.28}$$

When the contributions to a parameter from a history come from the terminations in the replaced process, the loss of weight at each step is the proper measure for the expected termination; in this case $\hat{I}_i$ for the ith history will be

$$\hat{I}_i = \sum_j W_{old,ij} (1-P_s) \cdot g(x_{ij}) \qquad (4.29)$$

where j denotes the jth occurrence of the replaced event in the ith history. $\hat{I}$ and S^2 are calculated as before using (4.27) and (4.28).

3. If the replaced process represents a decision between two branch points, the history must be split and followed from this point as two separate histories, one at each branch carrying a weight equal to the branching probability. To estimate I and S^2, formulas identical to (4.26), (4.27) and (4.28) can be used where the contributions from all daughter histories are considered.

Statistical Estimation

Statistical estimation can be implemented when applicable using the following steps.

1. Identify the stochastic process in the simulation which has the desired final outcome as one possible alternative.

2. When the process of interest is encountered in simulating a history, score the following contribution

$$g(\vec{X},Y_f)\ f(Y_f/\vec{X})$$

where $g(\vec{x},y)$ is the function being integrated by the simulation, Y_f is the desired final outcome of the process, $\vec{X}$ denotes the current state of all other variables in the system, and $f(Y_f/\vec{X})$ is the conditional probability of obtaining outcome Y_f given $\vec{X}$ as the status of the system.

3. The simulation should not be modified, but the stochastic process of interest is modeled by selecting Y from $(y/\vec{X})$.

4. If the outcome of step 3 is Y_f , no additional scoring is to be made. The contribution of this step remains $g(\vec{x}_i,Y_f)\ f(Y_f/\vec{X}_i)$.

5. Estimate the total contribution of history i , as

$$\hat{I}_i = \sum_j g(\vec{X}_{ij},Y_f)\ f(Y_f/\vec{X}_i) \qquad (4.30)$$

where j runs over all occurrences of the particular process being estimated in the ith history.

6. The estimation of I and S^2 can be calculated using (4.27) and (4.28).

Correlated Sampling

If there are two similar simulations involving only a single random variable and it is desired to estimate

$$\Delta = I_1 - I_2 \tag{4.31}$$

where

$$I_1 = \int_{-\infty}^{\infty} g_1(x) f_1(x) dx \tag{4.32}$$

$$I_2 = \int_{-\infty}^{\infty} g_2(y) f_2(y) dy \tag{4.33}$$

then correlated sampling can be implemented in this case as follows.

1. Generate N random numbers $R_1,\ldots,R_N$ from $U(0,1)$.
2. Generate a random sample $X_1,\ldots,X_N$ from $f_1(x)$ and another sample of the same size $Y_1,Y_2,\ldots,Y_N$ from $f_2(y)$ using

$$R_i = \int_{\infty}^{X_1} f_1(x) dx = \int_{-\infty}^{Y_1} f_2(y) dy \qquad i=1,\ldots N \tag{4.34}$$

3. Estimate Δ using

$$\hat{\Delta} = \frac{1}{N} \sum_{i=1}^{N} \Delta_i \tag{4.35}$$

where

$$\Delta_i = g_1(X_i) - g_2(Y_i) \tag{4.36}$$

4. Estimate the sample variance using

$$S^2 = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} \Delta_i^2 - \hat{\Delta}^2\} \tag{4.37}$$

Notice that the variance will be greatly reduced if $f_1(x)$ is similar to $f_2(y)$ and $g_1(x)$ is similar to $g_2(y)$, that is because the two random samples $X_1,\ldots,X_N$ and $Y_1,\ldots,Y_N$ will be highly correlated.

Control Variates

Control variates technique can be implemented using the following steps to evaluate the integral $I=\int g(x)f(x)dx$.

1. Express the parameter(s) to be estimated in integral form.
2. Obtain an approximate function $h(x)$ for each parameter I. The expected value of θ of $h(x)$ should be known.
3. Estimate the integral I_1,

$$I_1 = \int_{\infty}^{\infty} [g(x) - h(x)]\, f(x)dx \tag{4.38}$$

as

$$\hat{I}_1 = \frac{1}{N} \sum_{i=1}^{N} [g(X_i) - h(X_i)] \tag{4.39}$$

where $X_1,\ldots,X_N$ are a random sample generated from $f(x)$.

4. Calculate the final estimator of I as

$$\hat{I} = \theta + \hat{I}_1 \tag{4.40}$$

5. The variance of estimation can be calculated as

$$S^2 = \frac{N}{N-1} \{\frac{1}{N} \sum_{i=1}^{N} [g(X_i) - h(X_1)]^2 - \hat{I}_1^2\} \tag{4.41}$$

Example. The above steps were used to estimate the value of the integral given by (4.20). Again, the same integral is used to compare different techniques. The approximate function was chosen to be

$$h(x) = x \tag{4.42}$$

which has a mean value .5 at the range of integration (0-1) . The steps were coded in a computer program written

in FORTRAN V which is attached, with the results, in Appendix E. This method estimated the integral I given by (4.20) as .4255 with sample variance .0016 when the sample size was 20. It should be noted that the accuracy of estimation depends on how far the approximate function $h(x)$ mimics $f(x)$.

Antithetic Variates

To implement this technique, one should find two negatively correlated estimators for the parameter of interest. A linear combination of these two estimators can form a third estimator which would have a smaller variance than the variance of either of the original estimators. Steps to implement antithetic variates technique can be as follows:

1. Put the parameter to be estimated in integral form as

$$I = \int_{-\infty}^{\infty} g(x)\ f(x)dx \qquad (4.43)$$

2. Select a value for the parameter α between 0,1.
3. Generate a set of N random numbers from $U(0,1)$. denote them $R_1, R_2, \ldots, R_N$.
4. For each R_i , calculate two negatively correlated random variates X_i , X_i' . This can be accomplished using

$$\alpha R_i = \int_{-\infty}^{X_i} f(x)dx \tag{4.44}$$

$$1 - \alpha R_i = \int_{-\infty}^{X_i'} f(x)dx \tag{4.45}$$

5. Calculate the values of the new random variable,

$$\hat{\theta}_i = \alpha\, g(X_i) + (1-\alpha) g(X_i') \quad , \quad i=1,\ldots,N \tag{4.46}$$

6. Estimate I using

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i \tag{4.47}$$

7. The variance of estimation is

$$S^2 = \frac{N}{N-1} \{ \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i^2 - \hat{\theta}^2 \} \tag{4.48}$$

Notice that α can be simply taken equal to 1/2 or an iterative process can be used to determine the optimum value of α which gives the minimum sample variance.

Example. Considering again the evaluation of the integral I given by (4.20), one can use the above steps to implement antithetic variates technique as follows:

1. $I = \int_0^1 \frac{e^x - 1}{e - 1}\, dx$ (4.49)

2. $g(x) = \frac{e^x - 1}{e - 1}$ (4.50)

3. Let $\alpha = 1/2$ initially

4. Generate R_i from $U(0,1)$

5. $X_i = R_i$ (4.51)

6. $\hat{X}_i = 1 - R_i$ (4.52)

7. $\hat{\theta}_i = \alpha\, g(X_i) + (1-\alpha)\, g(\hat{X}_i)$ (4.53)

8. Repeat steps 4 through 7 for N times

9. Estimate I as

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i \quad (4.54)$$

The above steps were coded in a computer program which is attached in Appendix F. The value of α was changed in the range (.5-.95). The results are attached, also in Appendix F. From these results the accuracy of estimation appeared to be sensitive to the change of α .

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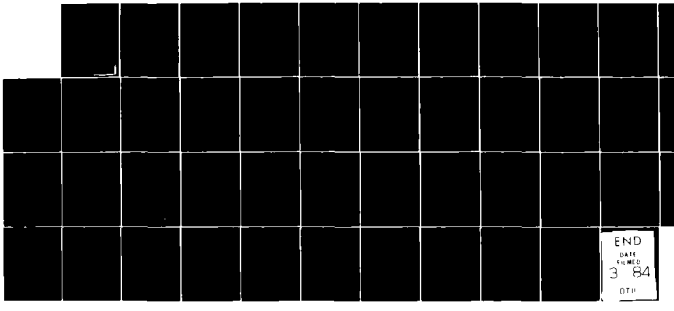

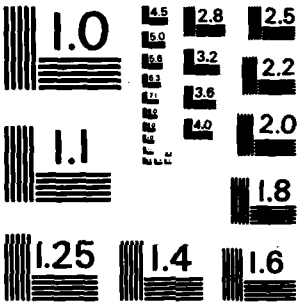

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Among the chosen values of α , the initial value, $\alpha = 1/2$, gives the minimum variance equal to .0004 which indicates that this method can achieve a great gain in the efficiency of estimation when compared with crude Monte Carlo with small additional computational effort. Notice that minimum variance is the criteria of accuracy of estimation in this case since the estimates resulting from using antithetic variates approach are unbiased.

V. Application of Variance Reduction Techniques

The earliest applications of VRTs were found in particle and radiation transport problems where very low probabilities are involved (20) during the 1940s and 1950s. In those days, the objective of applying VRTs was to compensate for the low speeds of computers which were frustrating when dealing with such problems (28). In the past two decades, computer speed has increased tremendously. This decreased the attention given to the development and application of VRTs for a while, then an increased demand for applying VRTs appeared due to the increasing complexity of problem simulations which would consume a great amount of computer time or result in reduced estimation accuracy if none of the VRTs were employed.

Recently, VRTs have found a wide application in almost all simulation areas: inventory simulation (6), queuing models simulation (10), network analysis (8), reliability studies, stationary (13) and non-stationary (14) simulation models, population growth, and simulation of Markov process (12). In this chapter, examples of these applications in the fields of inventory simulation, queuing simulation and computer performance measurement are presented. The aim of presenting these examples is to show how VRTs can be applied in such fields.

Application of Variance Reduction Technique for Inventory Simulation

Stockout in inventory control systems should be rare events. Therefore, variance reduction is an important experimental design issue to estimate accurately the mean values of those rare events in a simulation model.

Many VRTs are applicable to inventory models. Control variates, antithetic variates and conditional Monte Carlo methods are among those suitable techniques (6).

Ehrhardt (6) studied the use of each of those three techniques to reduce the variance in estimating the parameters of an inventory model. He combined two of them to achieve more reduction in variance of estimation. He concluded that conditional Monte Carlo is the best sole technique in reducing the variance when applied to the underlying inventory model. He also concluded that combining conditional Monte Carlo with either of the two other techniques would improve the variance reduction attained by the sole technique. Those conclusions were drawn from experimentation with the following inventory model.

The Inventory Model. A multi-item inventory system which is observable at discrete intervals of time was studied. Each of its items has an (iid) demand from one period to another. An order, when placed, is delivered after a fixed number of time periods L , and any unfulfilled demand in a period is backlogged to be satisfied later.

The inventory cost in the model consists of a fixed set up cost per order k, a holding cost per unit per period h, and a penalty cost per backlogged unit per period P.

Only partial knowledge of the demand distribution is available to the decision maker. Two policies, containing different degrees of information, were considered. Each of the two policies is of (s,S) type where s is the lower inventory replenishment level, and S is the upper inventory level.

The first policy, called the Empirical Normal Approximation policy, requires the knowledge of only the mean and the variance of the, assumed, normally distributed demand, M and σ^2. This policy can also be called the constant policy since s and S are considered constant in this case. In fact, expected values of the operating characteristics of the system can be calculated directly without simulation in this case using an analytical approach, but the author (6) used simulation only to compare this policy with the second policy.

The second policy considered is called Statistical Normal Approximation policy. In this case, only sample statistics of demand are available. The decision maker has to revise his policy periodically since he would not know that the demand distribution is stationary. During each revision interval, the sample mean and the variance of

demand are computed. These statistics are substituted for the actual mean and variance of the demand distribution to give an (s,S) policy to be used in the next interval where new statistics are collected. It was required to develop efficient simulation techniques to evaluate that policy.

The simulation experiment was designed to estimate mean values of the following five operating characteristics of the system.

1. Holding Quantity: the average number of units on hand at the end of a period.
2. Stockout Quantity: the average number of units backlogged at the end of a period.
3. Stockout Frequency: the fraction of periods in which demand is backlogged.
4. Ordering Frequency: the fraction of periods in which an order is placed.
5. Total Cost: the average total cost per period.

The objective of the study was to identify the suitable VRTs that yield low variance estimate of the expected values of the five operating characteristics for a given cost of computation.

Simulation Techniques. To estimate the expected values of the five operating characteristics, four simulation techniques were used; crude Monte Carlo (direct simulation), antithetic sampling, control variates and conditional Monte Carlo. When using each of these techniques, the vector

$X = \{x_i, i=1,2,\ldots,N\}$ represents the stochastic process of output of an N-period simulation of an operating characteristic. For any of the five operating characteristics, the expected value will be

$$\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i \tag{5.1}$$

Using the simulation method, j will estimate μ as $\mu_N^{(j)}$ for N periods.

Direct Simulation (Method 1). Using crude Monte Carlo the actual realizations are collected as simulated and is estimated as

$$\mu_N^{(1)} = \frac{1}{N} \sum_{i=1}^{N} X_i \tag{5.2}$$

the variance of this statistic is

$$\text{Var}\ \mu_N^{(1)} = \frac{1}{N^2} \sum_{i=1}^{N} \text{Var}\ X_i + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1, J\neq i}^{N} \text{Cov}(X_i, X_j) \tag{5.3}$$

Antithetic Sampling (Method 2). To apply the antithetic variates in this study, direct simulation was applied to the first $N/2$ periods, then the simulation was restarted where the antithetic variates were used to the second $N/2$ periods. That is, if the set U of uniform deviates, $\{U_i, i=1,2,\ldots,N/2\}$ is used to generate the first $N/2$ demands, then $\{(1-U_i), i=1,2,\ldots,N/2\}$ is the set of deviates used to

generate the second $N/2$ demands. The estimate of is

$$\mu_N^{(2)} = \frac{1}{N} \sum_{i=1}^{N/2} x_i + x_i' \tag{5.4}$$

where x_i' is the operating characteristic output when using the antithetic stream of deviates. The variance is given by

$$\text{Var}\ \mu_N^{(2)} = \frac{2}{N^2} \sum_{i=1}^{N/2} [\text{Var}\ x_i + \text{Cov}(x_i, x_i')]$$

$$+ \frac{2}{N^2} \sum_{i=1}^{N/2} \sum_{\substack{j=1 \\ i \neq j}}^{N/2} [\text{Cov}(x_i, x_j) + \text{Cov}(x_i, x_j')] \tag{5.5}$$

To achieve a reduction of the variance, the covariance terms in (5.5) should be sufficiently negative to make $\text{Var}_N^{(2)} < \text{Var}_N^{(1)}$ Which is not always true.

Control Variates (Methods 3a and 3b). To apply control variates approach, an approximate model should be simulated along with the model of interest. For a given operating characteristic $x^* = \{x_i^*, i=1,...N\}$ denotes the stochastic process of output for the approximate model simulation, and

$$\mu^* = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i^* \tag{5.6}$$

is the expected value of this output. μ^* is assumed to be known exactly. The control variates approach estimates μ as

$$\mu_N^{(3)} = \frac{1}{N} \sum_{i=1}^{N} \Gamma_i \tag{5.7}$$

where

$$\Gamma_i = x_i - \beta(x_i^* - \mu^*) \tag{5.8}$$

and β is a constant. The variance of $\mu_N^{(2)}$ is given by

$$\begin{aligned} \text{Var}\, \mu_N^{(3)} = & \frac{1}{N^2} \sum_{i=1}^{N} [\text{Var}\, x_i + \beta^2 \text{Var}\, x_i^* - 2\beta\, \text{Cov}(x_i, x_i^*)] \\ & + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} [\text{Cov}(x_i, x_j) + \beta^2\, \text{Cov}(x_i^*, x_j^*) \\ & - \beta\, \text{Cov}(x_i^*, x_j) - \beta\, \text{Cov}(x_i, x_j^*)] \end{aligned} \tag{5.9}$$

This variance depends on the chosen value of the constant β. It is difficult to determine the value of β that will minimize $\text{Var}\, \mu_N^{(3)}$. The study used two common approaches for the choice of β. Method 3a uses the regression estimate of β^* while β is set to be one in Method 3b. In both cases, the expected values of the operating characteristics from the Empirical Normal Approximation are control variates for the statistical policy simulation.

Conditional Monte Carlo (Methods 4a, 4b, 4c and 4d).

To apply the conditional Monte Carlo technique to the considered inventory model, the following notations have been used. Let $w = \{w_i, i=1,\ldots,N\}$ be a sequence of vectors specifying the state of the simulation at each period. Let $v = \{v_i, i=1,\ldots,N\}$ be a sequence of conditional expectations

$$v_i = E(x_i/w_i) \tag{5.10}$$

The estimate of μ is given in this case by

$$\mu_N^{(4)} = \frac{1}{N} \sum_{i=1}^{N} v_i \tag{5.11}$$

with a variance of estimation given by

$$\text{Var}\ \mu_N^{(4)} = \frac{1}{N^2} \sum_{i=1}^{N} \text{Var}\ v_i + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1, i\neq j}^{N} \text{Cov}(v_i, v_j) \tag{5.12}$$

The estimate of μ and the variance given by (5.11) and (5.12) can be used for each of the five operating characteristics considering the following notations. Let D and $D^{*(L+1)}$ be the demand in one period and in $(L+1)$ periods. Let Y_i denote inventory on hand and on order in period i and let s_i be the value of reorder policy in period i. Finally, let (a^+) be the max of a and 0. Using those notations, the conditional expectation for the holding

quantity is

$$V_i^{(1)} = E[(Y_i - D^{*(L+1)})^+] \tag{5.13}$$

and for stockout quantity

$$V_i^{(2)} = E[D^{*(L+1)} - Y_i)^+] \tag{5.14}$$

for stockout frequency

$$V_i^{(3)} = Pr\ (D^{*(L+1)} > Y_i) \tag{5.15}$$

for ordering frequency

$$V_i^{(4)} = Pr\ (D > (Y_i - s_i)) \tag{5.16}$$

and for total cost

$$V_i^{(5)} = h\ V_i^{(1)} + P\ V_i^{(2)} + k\ V_i^{(4)} \tag{5.17}$$

The functions given by (5.13) through (5.16) were calculated before simulation for a feasible range of arguments and then used appropriately in each simulation period as the conditional estimates.

Four variations of conditional Monte Carlo were examined in the study (6):

Method 4a: Conditional statistics are collected for all operating characteristics.

Method 4b: Conditional statistics are collected for all operating characteristics except ordering frequency which is simulated directly.

Method 4c: Conditional statistics are collected for operating characteristics except holding quantity which is simulated directly.

Method 4d: Conditional statistics are collected for only stockout quantity and stockout frequency.

In each of the above cases, total cost is taken as the weighted sum of the other characteristics.

Combined Methods (Methods 5a and 5b). When the results of Methods 1 through 4 were available, the author combined antithetic sampling and conditional Monte Carlo to get two mixed methods:

Method 5a: Method 4a with antithetic sampling

Method 5b: Method 4b with antithetic sampling

Summary of the Results

When simulation was performed for all mentioned methods, variance reductions gained by applying different VRTs compared with crude Monte Carlo technique were calculated. Antithetic sampling and control variates were found to yield meager results, while conditional Monte Carlo technique gave

superior performance. It reduced the variance of stockout quantity estimates by factors ranging from 1.0 to 7.6 for the statistical policy and from 5.4 to 740 for the constant policy. When conditional Monte Carlo was combined with antithetic variates method, the corresponding variance reduction ranges are 1.4 to 12 and 3.8 to 460, respectively. The power of the applied VRTs was also significant for estimates of the aggregate total cost. Specifically, conditional Monte Carlo reduced the variance of total cost by factors ranging from 2.1 to 14 for the statistical policy and from 2.3 to 480 for constant policy. When combined with antithetic sampling, the corresponding variance reduction ranges are 4.8 to 20 and 2.0 to 1300, respectively.

The author pointed out that the cost of computation was nearly the same for all variance reduction schemes, because the computational effort was dominated by updating the state of the system and by output analysis in this study. This means that the above given variance reduction factors can be a direct measure of the efficiency of corresponding VRT relative to crude Monte Carlo.

Application of VRTs in Queuing Problems

Another example of the fields of application of VRTs is the simulation of queuing problems. Many studies have been accomplished concerning the application of VRTs in this field. One of these studies (10) was done by Gaver where different Monte Carlo techniques were discussed and then applied to

queuing examples. The bases for straightforward Monte Carlo, antithetic variables, stratification, control variates, and concomitant variables methods were summarized in the paper The basis for the last method is worth mentioning since it has not been discussed before.

Concomitant Variables

Suppose that realizations of the random variables $\underline{X}$ (inter-arrival times, service times, etc.) are used to create the realization of W (waiting time of an individual in the queue), where

$$W^{(j)} = f(\underline{X}^{(j)}) \tag{5.18}$$

Commonly, $W^{(j)}$ and $X_i^{(j)}$ are monotonically related and then

$$\text{Cov}\,[W^{(j)}, X_i^{(j)}] = c_i \tag{5.19}$$

where c_i is either positive or negative. In fact,

$$E[X_i^{(j)}] = E[X_i] \tag{5.20}$$

since $\underline{X}$ is a given specified input. When sampling only k times, the realized X-values will deviate from their means. Then a linear correction to simple average will be needed

$$E[W]_c = \bar{W} + \sum_{i=1}^{I} \gamma_i \, (\bar{X}_i - E[X_i]) \qquad (5.21)$$

where γ_i can be estimated in terms of the covariance of W and X_i, and the resulting estimate is unbiased and consistent asymptotically as the sample size increases.

Queuing Examples

The VRTs mentioned before were well illustrated by consideration of a very simple queuing problem. The waiting time, w_n, of the n-th arrival to a single-server facility can be written as

$$w_n = \max [w_{n-1} - A_n + S_{n-1}, 0] \qquad (5.22)$$

where A_n is the inter-arrival period elapsing between the (n-1)th and n-th arrivals to the queue, and S_n is the service time of the n-th customer. If $\{A_n\}$ and $\{S_n\}$ are mutually independent sequences of (iid) random variables with $E[A_n] = E[A] > E[S_n] = E[S]$, and if other moments exist as required, then a stationary distribution for w_n exists as $n \to \infty$. The behavior of the system depends on the relation between $E[A]$ and $E[S]$; also, it depends on the number of arrivals n. The following cases were discussed by Gaver (10).

1. When $E[A]$ is little larger than $E[S]$, the queue tends to be long developing "heavy traffic" situation. In this case, approximate solution of the system based on diffusion equation solution is available.
2. When $E[A] < E[S]$ the queue tends to grow and little information is available. In this case, (5.22) will be simply

$$w_n^* = w_{n-1}^* - A_n + S_{n-1} \tag{5.23}$$

where w_n^* will be approximately normal if A_n and S_n have finite variances. The mean of w_n in this case is

$$E[w_n] = (n-1)(E[S] - E[A]) \ , \ w_1 = 0 \tag{5.24}$$

For small and moderate values of n, the variances of A and S may not be finite, then one would have to use simulation to estimate $E[w_n]$.

The author (10) applied various VRTs to estimate $E[w_n]$ for selected values of n, focusing on control variates and concomitant variables approaches. The following numerical example was used to display the effect of selected VRTs on the accuracy of estimation.

Numerical Example. In a single server queuing system, service times are taken to be exponentially distributed (with mean $\mu^{-1} = 10/9$) and the interarrival times of customers are taken to be constant (regular arrivals) or exponentially distributed (poisson fashion arrivals), with unity mean in both cases.

In spite of the apparent simplicity of such a system, its transient response is not easily characterized mathematically. Simulation should be used to estimate the parameters of the system or, alternatively, the diffusion approximation would be used. Results of applying the selected VRTs and diffusion approximation are tabulated in Tables 5.1 and 5.2 for the regular and poisson arrival cases. Discussion of the results in those tables is given below.

Discussion of Results

Rows (1) and (2) of Tables 5.1 and 5.2 show the results obtained when 25 independent realizations were averaged to estimate $E[w_n]$. The same random numbers were used to apply antithetic variates approach to estimate $\bar{w}_n(a)$ and its variance, rows (3) and (4). Comparison of variances in rows (2) and (4) indicates that antithetic variates approach has produced an improvement even after considering the additional computational effort when simulating a total of 50 realizations. The improvement in Table 5.2 is smaller than that in Table 5.1 due to the added variability contributed by the random arrivals.

The simple control variable technique, rows (6) and (7), gave better improvement for large values of n than antithetic variates approach. This appears clearly in Table 5.2. Rows (8) and (9) display the effect of adjusting the straightforward estimate in accordance with the concomitant variable that equals the sum of the first n service times in Table 5.1, while Table 5.2 considers both service and arrival times as concomitant variables. The behavior of the concomitant variables technique was similar to the control variables and antithetic variables. Rows (9) and (10) display the results of applying concomitant variables to the components of the antithetic estimate of (3) and (4). Results in this case are better than any of the above cases. Rows (12) and (13) indicate the value of regression-adjusted control procedure where regression was used to determine the value of $\hat{B}_o$ used in the modified estimator of the mean waiting time.

$$E[\tilde{w}] = \bar{w} + \hat{B}_o \, (\bar{w}^* - E[w^*]) \tag{5.25}$$

Row (4) shows the estimation of the mean of w_n when diffusion approximation was used. The results in row (14) agree quite closely with control variables, row (6), and regression-adjusted estimates, rows (8) and (10), for large n .

Table 5.1. Estimated Wait of n-th Customer

$A_n = 1$, $ES_n = 1.111$ (Regular Arrivals, Exponential Service at Single Server) Based on 25 Realizations

			n: 5	10	25	50	100	150	200
Straightforward:	(1)	Mean	1.19	1.87	4.51	7.85	11.43	17.04	23.60
	(2)	Variance	.124	.128	.605	1.356	2.758	6.795	10.673
Antithetic:	(3)	Mean	1.35	2.51	5.11	8.67	14.50	19.58	24.88
	(4)	Variance	.054	.141	.256	.367	.622	1.370	1.770
Random Walk:	(5)	Mean	.44	.99	2.67	5.44	11.00	16.55	22.11
Control: (β=1)	(6)	Mean	1.48	2.91	5.92	9.86	16.01	22.12	28.00
	(7)	Variance	.039	.102	.373	.763	.963	1.296	1.427
Straightforward & regression:	(8)	Mean	1.42	2.27	5.49	9.48	14.62	22.50	29.08
	(9)	Variance	.019	.042	.144	.366	.446	1.715	3.864
Antithetic & regression:	(10)	Mean	1.38	2.10	5.16	8.78	15.01	21.72	27.46
	(11)	Variance	.014	.106	.165	.093	.093	.671	1.350
Control:	(12)	Mean	1.39	2.43	5.55	9.05	14.54	20.94	27.13
(β estimated)	(13)	Variance	.016	.034	.126	.198	.445	.733	.832
Diffusion (Asymptotic; (5)+Constant:	(14)	Mean	5.9	6.5	8.2	10.9	16.5	22.1	27.6

Table 5.2. Estimated Expected Wait of n-th Customer

$EA_n = 1$ $ES_n = 1.111$ (Poisson Arrivals, Exponential Service at Single Server) Based on 25 Realizations

			n: 5	10	25	50	100	150	200
Straightforward:	(1)	Mean	2.15	2.77	6.92	11.93	19.84	25.65	29.28
	(2)	Variance	.225	.295	1.119	1.832	2.692	6.596	11.446
Antithetic:	(3)	Mean	1.95	3.33	6.62	11.22	17.30	24.33	29.55
	(4)	Variance	.062	.107	.271	.558	1.079	1.964	3.239
Random Walk:	(5)	Mean	.44	.99	2.67	5.44	11.00	16.55	22.11
Control:	(6)	Mean	1.63	3.05	6.13	9.81	15.70	21.55	27.23
(β =1)	(7)	Variance	.125	.225	.477	1.026	1.278	1.396	1.393
Straightforward & regression:	(8)	Mean	1.69	2.86	6.04	10.79	17.93	22.46	27.64
	(9)	Variance	.148	.087	.162	.443	1.201	1.565	2.307
Antithetic & regression:	(10)	Mean	1.73	3.22	6.15	10.83	17.61	24.09	29.57
	(11)	Variance	.025	.054	.074	.216	.368	1.016	.974
Control:	(12)	Mean	1.84	2.92	6.41	10.65	17.03	22.38	27.41
(β estimated)	(13)	Variance	.051	.078	.196	.397	.866	1.040	1.297
Diffusion (Asymptotic; (5)+constant:	(14)	Mean	10.4	11.0	12.7	15.4	21	26.6	32.1

Application of VRTs in Computer System Performance Measurement

Another example of fields of applications of VRTs is computer system performance measurement. A study in this field (7) was performed where the antithetic variates approach was employed to reduce the variance of estimation.

Performance evaluation of a computer system is very important in both design and utilization phases. In the second case, the most important and difficult problem in the design of a measurement experiment is the determination of its time duration since the increase of this duration increases the cost of the experiment. On the other hand, the time duration of the measurement experiment should be long enough to yield a good estimate of the unknown parameters. This contradiction is similar to that which arises in the simulation problems where the sample size utilized should be large enough to obtain a good estimate of the unknown population means and, at the same time, this sample size should be small enough to keep the simulation cost feasible.

The above problem was solved in simulation by utilization of the suitable VRTs which, when applicable, give the required precision using a small sample size. This fact encourages the use of such techniques in the field of computer performance measurement. To examine the profitability of applying VRT in this field, an experiment with an existing synthetic job generator (described later) for the computer system was designed and performed using antithetic variates as a VRT.

The Synthetic Job Generator

The synthetic job generator used in this experiment was designed to model the I/O (Input/Output) behavior of user jobs in a CDC 6400 computer system. The values of the parameters of this generator were measured by employing a trace technique which records a complete history of CPU (central processing unit) containing the time switched between user jobs and a history of the two disks. The synthetic job generator was designed according to the information obtained from tracing the real system for a period of eight hours of normal production processing. Then the designed synthetic job generator was used to perform the following experiment.

The Experiment. A single performance measure for the system, namely the mean job elapsed time (i.e., the mean time a job generated by the synthetic job generator spends in the system) was used. The population is composed of the elapsed times of all possible jobs the generator may produce. Its mean is to be estimated using a small sample size. Let $\bar{t}_n$ be the mean of a sample of size n. If many samples of the same size were drawn from the population, then $\bar{t}_n$ will be a random variable with a mean denoted by $E(\bar{t}_n)$ and a variance denoted by $Var(\bar{t}_n)$. To use the method of antithetic variates, two samples of $N/2$ jobs each are used. Let $\bar{t}_{N/2}$ be the mean of the first, and $\bar{t}'_{N/2}$ be the mean of the second. The antithetic estimator of the population mean is

$$\bar{t}_n^* = \alpha\, t_{n/2} + (1 - \alpha)\, \bar{t}'_{n/2} \quad , \quad 0 < \alpha < 1 \tag{5.26}$$

and the variance of $\bar{t}_n^*$ is given by

$$\mathrm{Var}(\bar{t}_n^*) = \alpha^2\, \mathrm{Var}(\bar{t}_{n/2}) + (1 - \alpha)^2 \mathrm{Var}(\bar{t}'_{n/2})$$

$$+ \; 2\alpha(1 = \alpha)\, \mathrm{Cov}(\bar{t}_{n/2}, \bar{t}'_{n/2}) \tag{5.27}$$

To establish the negative correlation between the random variable realizations in the two samples, the random variable values in the first sample is generated using a sequence of random numbers $R_1, R_2, \ldots, R_{n/2}$ from $U(0,1)$ and those of the second sample are generated using $(1-R_1), (1-R_2), \ldots (1-R_N)$. In this experiment, n was chosen to be 80 jobs, 40 jobs for each sample, and α was chosen to be 1/2. When the obtained results were analyzed, the authors drew the following conclusions.

Conclusions. The authors concluded that their experiment to demonstrate the feasibility of applying antithetic variates method was not completely successful, but they also concluded that this lack of success was due to the instability of the devised synthetic job generator they used. It is the opinion of the authors that stochastic job generators can be designed which have all the properties required to make the method of antithetic variables very effective in reducing the

variance of an estimator and, hence, for a given confidence interval and a given confidence level, in decreasing the minimum duration of a measurement run. The authors also stated that better results would have been obtained if values less than 1/2 had been chosen for α .

Comment. It is clear from the authors' conclusions that the application of the antithetic variates method in their experiment was not completely successful for the reasons given above. It would be better if they tried to use other values of α to minimize the variance for a given status of the experiment. They could also have obtained better results if their synthetic job generator was more stable and had the features suitable for applying antithetic variates approach. Another possibility would be to try other applicable VRTs.

VI. Conclusions and Recommendations

VRTs were investigated, clarified, contrasted and illustrated in the former chapters. From the extensive literature review and the computed numerical examples, it was found that none of the known VRTs is generally superior to the others. In other words, there is no single technique which is the most suitable technique for every simulation problem. The condensed table (Table 3.2) of VRTs' characteristics identified the fields of application of each technique. It is clear that various techniques can be applied to the same problem in certain fields. In this case, one can compare the efficiency of various techniques. The most efficient technique is the one which utilizes most of the available information about the underlying simulation process and gives the most accurate estimation of the parameters (minimizes the sample variance) with minimum computational effort. In many cases, these three extreme objectives cannot be achieved simultaneously by one of the VRTs. Only the involved analyst can weigh them to choose the optimal technique suitable for his simulation problem, objectives, and available resources. It is not practical that an analyst, handling a real world problem, will apply several applicable VRTs to his problem and then compare them to choose the optimal technique. A study like this and other studies devoted to the application of VRTs to the specific kinds of

problems can help the analyst select a profitable VRT. The degree of profitability will then depend on how successfully the technique is implemented.

A possible useful extension of this work is an organized collection of real world problems in different fields of applications.

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APPENDIX A

Computer Program Listing with the Results for the Importance Sampling Example

```
      PROGRAM IMPORT
C     ***************************************************
C     THIS PROGRAM IS AN EXAMPLE OF USING IMPORTANCE SAMPLING
C     ***************************************************
      REAL *1,F2,TERM1,TERM2
      WRITE(*,150)
      DATA N /20/
      SUM1=0.0
      SUM2=0.0
      WRITE(*,90)
   90 FORMAT(10X,' R',9X,' X ',9X,'F1',9X,'F2',9X,'TERM1')
      WRITE(*,150)
      DO 100 I=1,N
      CALL RAND (R)
      X = ALOG(1/(1-R))
      F1=FAX(X)
      F2 = FAG(X)
      TERM1=F1/F2
      TERM2= TERM1**2
      IF(X.LT.1.0)THEN
      SUM1=SUM1+TERM1
      END IF
      SUM2=SUM2+TERM2
  100 WRITE (*,110) R,X,F1,F2,TERM1
  110 FORMAT(5X,5(F9.6,3X))
      PAR=SUM1/N
      VAR=N/(N-1)*(SUM2/N-PAR**2)
      WRITE(*,150)
  150 FORMAT(8X,55('*'))
      WRITE(*,120) PAR,VAR
  120 FORMAT(9X,'ESTIMATED PAR = ',F9.7,'  VARIANCE = ',F9.7)
      WRITE(*,150)
      END
      FUNCTION FAX(Y)
      FAX=.01*EXP(-.01*Y)
      RETURN
      END
      FUNCTION FAG(Y)
      FAG=EXP(-Y)
      RETURN
      END
      SUBROUTINE RAND(R)
      REAL R,S,XM
      INTEGER IS
      DATA S,XM /.23979,923.534/9/
      S=S*XM
      IS=S
      R=S-IS
      S=R+.8234E17
      RETURN
      END
```

..RUN,FTN5

****

R	X	F1	F2	TERM1
.467170	.629552	.009937	.532830	.018650
.879721	2.117943	.009790	.120279	.081398
.630365	.995240	.009901	.369635	.026786
.276951	.324279	.009968	.723049	.0137[illegible]
.228132	.258942	.009974	.771868	.01292[illegible]
.024116	.024411	.009998	.975884	.010241
.009957	.010007	.009999	.990043	.010100
.349598	.430164	.009957	.650402	.01530[illegible]
.055349	.056940	.009994	.944651	.010580
.731006	1.313066	.009270	.268994	.036671
.158201	.172214	.009983	.841799	.011859
.433205	.567757	.009943	.566795	.017043
.908567	2.392154	.009764	.091433	.106785
.386255	.488176	.009951	.613745	.016214
.243985	.279694	.009972	.756015	.013190
.079153	.082461	.009992	.920847	.010051
.334626	.407406	.009959	.665374	.014969
.725597	1.293159	.009872	.274403	.035975
.704083	1.217676	.009879	.295917	.03338[illegible]
.986214	4.284094	.009581	.013786	.6949[illegible]

****

ESTEMATED PAR = .0101501 VARIANCE = .0252940

****

61300 CM STORAGE USED.
0.129 CP SECONDS COMPILATION TIME.
END IMPORT
15200 MAXIMUM EXECUTION FL.
0.054 CP SECONDS EXECUTION TIME.

APPENDIX B

Computer Program Listing with the Results for Russian Roulette Example, Case 1

```
      PROGRAM RSLIB
C     ****************************************************************
C     THIS PROGRAM IS AN EXAMPLE OF USING RUSSIAN ROULETTE
C        THE SYSTEM CONSISTS OF THREE SUBSYSTEM IN SERIES
C     ****************************************************************
      REAL P1(3),P2(3),V(3)
      NS=1000
      N=20
      DATA (P1(I),I=1,3) /.7,.5,.2/
      PX=.25
      WRITE(*,35)
      DO 200 L=1,10
      PX=PX-.05
      NRDC=0
      ICOUNT=0
      WRITE(*,25)PX,(P1(I),I=1,3)
   25 FORMAT(/,10X,'PX = ',F4.3,5X,'P1 =  ',F4.3,2X,'P2 = ',F4.3,2
X,'P3='
     + ,F4.3)
      DO 10 I=1,3
   10 V(I)=P1(I)*(1-P1(I))/N
      DO 100 J=1,1000
      CALL NORM(RN)
      X1=P1(1)+RN*SQRT(V(1))
      IF(X1.LT.PX)THEN
      ICOUNT=ICOUNT+1
      NRDC=NRDC+2
      GO TO 100
      ELSE
      CALL NORM(RN)
      X2=P1(2)+RN*SQRT(V(2))
      R12=X1*X2
      IF(R12.LT.PX) THEN
      ICOUNT=ICOUNT+1
      NRDC=NRDC+3
      GO TO 100
      ELSE
      CALL NORM(RN)
      X3=P1(3)+RN*SQRT(V(3))
      RS=R12*X3
      IF(RS.LT.PX)THEN
      ICOUNT=ICOUNT+1
      END IF
      END IF
      END IF
  100 CONTINUE
      CI=ICOUNT/1000.0
      CL=1-CI
```

```
      WRITE(*,15)
   15 FORMAT (/,10X,50('*'))
      WRITE(*,20) NRDC,CL
   20 FORMAT(10X,'NO. OF REDUCTIONS =',I5,3X,'P(RS GT RY ) =',3X,
F6.4)
  200 WRITE (*,15)
      END
      SUBROUTINE NORM(RN)
      A=0.0
      C1=.029899776
      C2=.008255968
      C3=.076542912
      C4=.252408784
      C5=3.949846138
      DO 4 I=1,12
      CALL RAND(R)
    4 A=A+R
      RX=(A-6.)/4
      R2=RX*RX
      RN=((((C1*R2+C2)*R2+C3)*R2+C4)*R2+C5)*RX
      RETURN
      END
      SUBROUTINE RAND(R)
      REALR,S,XM
      INTEGER IS
      DATA S,XM/.23978,823.53470/
      S=S*XM
      IS=S
      R=S-IS
      S=R+.8234817
      RETURN
      END
```

```
RUN,PTNS
     ,1300 CM STORAGE USED.
     0.173 CP SECONDS COMPILATION TIME.

        ********************************************************

        PX = .300   P1 = .700   P2 = .800   P3=.800

        ********************************************************
        NO. OF REDUCTIONS =     1   P(RS GT PX ) =     .9514

        ********************************************************

        PX = .350   P1 = .700   P2 = .800   P3=.800

        ********************************************************
        NO. OF REDUCTIONS =    17   P(RS GT PX ) =     .8700

        ********************************************************

        PX = .400   P1 = .700   P2 = .800   P3=.800

        ********************************************************
        NO. OF REDUCTIONS =    45   P(RS GT PX ) =     .6530

        ********************************************************

        PX = .450   P1 = .700   P2 = .800   P3=.800

        ********************************************************
        NO. OF REDUCTIONS =   160   P(RS GT PX ) =     .4730

        ********************************************************

        PX = .500   P1 = .700   P2 = .800   P3=.800

        ********************************************************
        NO. OF REDUCTIONS =   366   P(RS GT PX ) =     .2500

        ********************************************************

        PX = .550   P1 = .700   P2 = .800   P3=.800

        ********************************************************
        NO. OF REDUCTIONS =   542   P(RS GT PX ) =     .1457

        ********************************************************

        PX = .600   P1 = .700   P2 = .800   P3=.800

        ********************************************************
```

```
     NO. OF REDUCTIONS =  823   P(RS GT PX ) =     .0530

     ***********************************************************

     PX = .650   P1 = .750   P2 = .800   P3=.300

     ***********************************************************
     NO. OF REDUCTIONS =  1122   P(RS GT PX ) =     .0240

     ***********************************************************

     PX = .700   P1 = .750   P2 = .800   P3=.300

     ***********************************************************
     NO. OF REDUCTIONS =  1430   P(RS GT PX ) =     .0000

     ***********************************************************

     PX = .750   P1 = .750   P2 = .800   P3=.300

     ***********************************************************
     NO. OF REDUCTIONS =  1654   P(RS GT PX ) =     .0000

     ***********************************************************
END  FILE
15200 MAXIMUM EXECUTION FL.
3.722 CP CPU TIME.
```

APPENDIX C

Computer Program Listing with the Results for Russian Roulette Example, Case 2

```
      PROGRAM RELI
C     ******************************* [illegible]
C     THIS PROGRAM [illegible]
C     THE SYSTEM CONSISTS OF THREE SUBSYSTEMS IN PARALLEL
C     ******************************* [illegible]
      REAL P1(3),[illegible],V(3)
      NS=1000
      N=30
      DATA (P1(I),I=1,3) /.7,.7,.7/
      PX=.9
      WRITE(*,15)
      PRINT *,' THREE SUBSYSTEMS IN PARALLEL'
      DO 200 L=1,10
      PX=PX+.005
      NRDC=0
      ICOUNT=0
      WRITE(*,25)PX,(P1(I),I=1,3)
   25 FORMAT(/,10X,'PX = ',F4.2,2X,'P1 = ',F4.3,2X,'P2 = ',F4.3,2X
X,'P3='
     + ,F4.3)
      DO 10 I=1,3
   10 V(I)=P1(I)*(1-P1(I))/N
      DO 100 I=1,1000
      CALL NORM(RN)
      X1=P1(1)+RN*SQRT(V(1))
      IF(X1.GT.PX)THEN
      ICOUNT=ICOUNT+1
      NRDC=NRDC+2
      GO TO 100
      ELSE
      CALL NORM(RN)
      X2=P1(2)+RN*SQRT(V(2))
      R12=1-(1-X1)*(1-X2)
      IF(R12.GT.PX) THEN
      ICOUNT=ICOUNT+1
      NRDC=NRDC+1
      GO TO 100
      ELSE
      CALL NORM(RN)
      X3=P1(3)+RN*SQRT(V(3))
      RS=1-(1-X1)*(1-X2)*(1-X3)
      IF(RS.GT.PX)THEN
      ICOUNT=ICOUNT+1
      END IF
      END IF
      END IF
  100 CONTINUE
      C1=ICOUNT/1000.0
```

```
      WRITE(*,15)
   15 FORMAT (' ',10X,50('*'
      WRITE(*,20) NRDC,CI
   20 FORMAT(10X,'NO. OF REDUCTIONS =',I5,3X,'(FRS ST RX ) =',3X,
F6.4)
  200 WRITE (*,15)
      END
      SUBROUTINE NORM(RN)
      A=0.0
      C1=.029899776
      C2=.008355968
      C3=.076542912
      C4=.252408784
      C5=3.949940138
      DO 4 I=1,12
      CALL RAND(R)
    4 A=A+R
      RX=(A-6.)/4
      R2=RX*RX
      RN=((((C1*R2+C2)*R2+C3)*R2+C4)*R2+C5)*RX
      RETURN
      END
      SUBROUTINE RAND(R)
      REALR,S,XM
      INTEGER IS
      DATA S,XM/.23978,823.53478/
      S=S*XM
      IS=S
      R=S-IS
      S=R+.8234817
      RETURN
      END
```

```
RUN,FTN5
     61300 CM STORAGE USED.
     0.179 CP SECONDS COMPILATION TIME.

        **************************************************
       THREE SUBSYSTEMS IN PARALLEL

        PX = .905   P1 = .700   P2 = .700   P3=.700

        **************************************************
        NO. OF REDUCTIONS =  573   P(RS GT PX ) =   1.0000

        **************************************************

        PX = .910   P1 = .700   P2 = .700   P3=.700

        **************************************************
        NO. OF REDUCTIONS =  555   P(RS GT PX ) =   1.0000

        **************************************************

        PX = .915   P1 = .700   P2 = .700   P3=.700

        **************************************************
        NO. OF REDUCTIONS =  497   P(RS GT PX ) =    .9990

        **************************************************

        PX = .920   P1 = .700   P2 = .700   P3=.700

        **************************************************
        NO. OF REDUCTIONS =  416   P(RS GT PX ) =    .9980

        **************************************************

        PX = .925   P1 = .700   P2 = .700   P3=.700

        **************************************************
        NO. OF REDUCTIONS =  394   P(RS GT PX ) =    .9950

        **************************************************

        PX = .930   P1 = .700   P2 = .700   P3=.700

        **************************************************
        NO. OF REDUCTIONS =  298   P(RS GT PX ) =    .9920

        **************************************************

        PX = .935   P1 = .700   P2 = .700   P3=.700
```

```
**************************************************
NO. OF REDUCTIONS =  253   P(RS GT PX ) =    .9820

**************************************************

PX = .940   P1 = .700   P2 = .700   P3=.700

**************************************************
NO. OF REDUCTIONS =  209   P(RS GT PX ) =    .9770

**************************************************

PX = .945   P1 = .700   P2 = .700   P3=.700

**************************************************
NO. OF REDUCTIONS =  151   P(RS GT PX ) =    .9710

**************************************************

PX = .950   P1 = .700   P2 = .700   P3=.700

**************************************************
NO. OF REDUCTIONS =  106   P(RS GT PX ) =    .9360

**************************************************
END  RELIB
16000 MAXIMUM EXECUTION FL.
4.323 CP SECONDS EXECUTION TIME.
```

APPENDIX D

Computer Program Listing with the Results for the Examples of Systematic Sampling and Stratified Sampling

```
L.A

100=       PROGRAM MASTER
101=C      ***************************************************************
110=C       THIS PROGRAM IS AN EXAMPLE OF USING,
120=C            SYSTEMATIC SAMPLING,
130=C            STRATIFIED SAMPLING  AS  U.I.T'S.
140=C         IT ALSO PERFORMS CRUDE MONTE CARLO TECHNIQUE TO COMPARE
150=C         THE RESULTS OF THE THREE METHODS.
160=C        ***************************************************************
170=C        NOTE :
180=C           METHOD = 1          CRUDE MONTE CARLO
190=C                    2          SYSTEMATIC SAMPLING
191=C                    3         STRATIFIED SAMPLING
200=       INTEGER  I,J,K,N,NS(10),METHOD
210=       REAL    AL(10),AB(10),AC(10),X,Y,FY,TERM,SMF,DS,FUNC
220=       DATA N,K,METHOD /16,4,2/
230=       FM=0.0
240=       DO 1 I=1,50
250=     1 CALL RANB (R)
260=       IF(K.EQ.1)THEN
270=       WRITE(*,3)
280=     3 FORMAT(20X,'CRUDE MONTE CARLO ')
290=       NS(1)=N
300=       GO TO 10
310=       END IF
320=       DATA (NS(J),J=1,4) /4,4,4,4/
330=       IF(METHOD.EQ.2)THEN
340=       WRITE(*,5)
350=     5 FORMAT(20X,' SYSTEMATIC SAMPLING ')
360=       GO TO 10
370=         ELSE
380=       WRITE(*,6)
390=     6 FORMAT(20X,'STRATIFIED SAMPLING')
400=       DATA(AB(J),J=1,4) /0.0,.36,.62,.83/
410=       DATA(AL(J),J=1,4) /.36,.62,.83,1.0/
420=       GO TO 110
430=       END IF
440=    10 DS=1.0/K
450=       DO 100 J=1,K
460=       AL(J)=J*DS
470=   100 AB(J)=AL(J)-DS
480=   110 DO 120 J=1,K
490=   120 AC(J)=AL(J)-AB(J)
500=       WRITE(*,130)
510=   130 FORMAT(3X,60('*'))
520=       WRITE(*,140)
530=   140 FORMAT(6X,'J',3X,'J',5X,'X',10X,'Y',11X,'FY',6X,'TERM',7Y,'
FM')
```

```
      WRITE(*,130)
      SMF=0.0
      SM1=0.0
      SM2=0.0
      DO 205 J=1,K
      SM1=0.0
      DO 200 I=1,NS(J)
  150 CALL RAND (RN)
      IF (RN.GT.AL(J).OR.RN.LE.AB(J)) THEN
      GO TO 150
      END IF
      X=RN
      Y=AB(J)+AC(J)*X
      FY=FUNC(Y)
      SM1=SM1+FY
      FY2=FY**2
      WRITE(*,160)J,I,X,Y,FY
  160 FORMAT (3X,2I4,3(F8.6,3X))
  200 SM2=SM2+FY2
      TERM=AC(J)*SM1/NS(J)
      FM=FM+TERM
      WRITE(*,201)TERM,FM
  201 FORMAT (43X,2(F8.6,3X))
  205 TERM2=(SM2*AC(J)**2-NS(J)*TERM**2)/(NS(J))
      PAR=FM
      VAR=TERM2
      WRITE(*,130)
      WRITE(*,220) PAR,VAR
  220 FORMAT(3X,'ESTIMATE OF PAR =',F8.6,5X,' VARIANCE =',F8.6)
      WRITE(*,130)
      END
      FUNCTION FUNC(Y)
      REAL FY,Y
      FUNC=(EXP(Y)-1.0)/1.718
      FY=FUNC
      RETURN
      END
      SUBROUTINE RAND(R)
      REAL R,S,XM
      INTEGER IS
      DATA S,XM /.23978,823.53478/
      S=S*XM
      IS=S
      R=S-IS
      S=R+.8234617
      RETURN
      END
```

```
 /16,4,3/=/16,1,1/,A
         1 CHANGE(S)
..RUN,FTN5
      61300 CM STORAGE USED.
      0.194 CP SECONDS COMPILATION TIME.



                  CRUDE MONTE CARLO
************************************************************
  J    I      X           Y           FY        TERM      FM
************************************************************
  1    1 .209920     .209920     .135960
  1    2 .025423     .025423     .014988
  1    3 .085874     .085874     .052194
  1    4 .869429     .869429     .806489
  1    5 .154521     .154521     .097264
  1    6 .402926     .402926     .288822
  1    7 .972634     .972634     .957451
  1    8 .147175     .147175     .092292
  1    9 .353352     .353352     .246701
  1   10 .146870     .146870     .092086
  1   11 .101697     .101697     .062309
  1   12 .900027     .900027     .849633
  1   13 .352683     .352683     .246146
  1   14 .595825     .595825     .474113
  1   15 .831571     .831571     .754904
  1   16 .977236     .977236     .964552
                                            .383494   .383494
************************************************************
 ESTIMATE OF FAR = .383494      VARIANCE = .119850
************************************************************
      END  MASTER
      13300 MAXIMUM EXECUTION FL.
      0.043 CP SECONDS EXECUTION TIME.
FILE QUOTA EXCEEDED
..
```

```
RUN,FTN5
     61300 CM STORAGE USED.
     0.195 CP SECONDS COMPILATION TIME.
                          SYSTEMATIC SAMPLING
*********************************************************************
   J    I      X            Y            FY          TERM        FM
*********************************************************************
   1    1 .209920      .052480      .031363
   1    2 .025423      .006356      .003711
   1    3 .085874      .021468      .012631
   1    4 .154521      .038630      .022926
                                                  .004414     .004414
   2    1 .402926      .350731      .244532
   2    2 .353352      .338338      .234351
   2    3 .352683      .338171      .234214
   2    4 .386681      .346670      .241182
                                                  .059642     .064057
   3    1 .594441      .648610      .531362
   3    2 .691941      .672986      .558836
   3    3 .537404      .634351      .515598
   3    4 .720287      .680072      .566950
                                                  .135797     .199853
   4    1 .938755      .984689      .976122
   4    2 .806858      .951715      .925560
   4    3 .901946      .975486      .961849
   4    4 .933105      .983276      .973923
                                                  .239842     .439696
*********************************************************************
 ESTIMATE OF PAR = .439696       VARIANCE = .022076
*********************************************************************
     END  MASTER
     13300 MAXIMUM EXECUTION FL.
     0.050 CP SECONDS EXECUTION TIME.
FILE QUOTA EXCEEDED
..
```

```
RUN,FTN5
     61300 CM STORAGE USED.
     0.178 CP SECONDS COMPILATION TIME.
                    STRATIFIED SAMPLING
*****************************************************************
   J    I      X             Y             FY        TERM         FM
*****************************************************************
   1    1 .209920      .075571     .045693
   1    2 .025423      .009152     .005352
   1    3 .085874      .030915     .018274
   1    4 .154521      .055628     .033297
                                               .009235      .009235
   2    1 .402926      .464761     .344373
   2    2 .595825      .514914     .392023
   2    3 .386681      .460537     .340468
   2    4 .594441      .514555     .391672
                                               .095455      .104690
   3    1 .691942      .765506     .669183
   3    2 .755606      .778677     .686024
   3    3 .822945      .792819     .704084
   3    4 .720287      .771260     .676653
                                               .143637      .248327
   4    1 .938755      .989588     .983776
   4    2 .901946      .983331     .974008
   4    3 .933105      .988628     .982273
   4    4 .962616      .993679     .990194
                                               .167036      .415363
*****************************************************************
ESTIMATE OF PAR = .415363       VARIANCE = .017465
*****************************************************************
     END  MASTER
     13300 MAXIMUM EXECUTION FL.
     0.044 CP SECONDS EXECUTION TIME.
FILE QUOTA EXCEEDED
..
```

APPENDIX E

Computer Program Listing with the Results for the Example of Control Variates

```
L,A

100=      PROGRAM CONTRL
110=C     ************************************************
120=C      THIS PROGRAM USES CONTROL VARIATE AS V.R.T.
130=C     ************************************************
140=      INTEGER I,N
150=      REAL TERM1,TERM2,SMT1,SMT2,FAX,FUNC
160=      DATA N,FI /20,.5/
170=      SMT1=0.0
180=      SMT2=0.0
190=      SMT3=0.0
200=      WRITE(*,10)
210=   10 FORMAT(7X,50('*'))
220=      WRITE(*,20)
230=   20 FORMAT(10X,'RN',12X,'TERM1',7X,'TERM2',8X,'TERM3')
240=      WRITE(*,10)
250=      DO 100 I=1,N
260=      CALL RAND(RN)
270=      TERM1=FUNC(RN)
280=      SMT1=SMT1+TERM1
290=      TERM2=FAX(RN)
300=      TERM3=(TERM1-TERM2)**2
310=      WRITE (*,50)RN,TERM1,TERM2,TERM3
320=   50 FORMAT(5X,4(4X,F9.6))
330=      SMT3=SMT3+TERM3
340=  100 SMT2=SMT2+TERM2
350=      T1=SMT1/N
360=      T2=SMT2/N
370=      FMEAN=T1-T2+FI
380=      VAR=(SMT3-N*(T1-T2)**2)/(N-1)
390=      WRITE(*,10)
400=      WRITE(*,109)
410=  109 FORMAT(11X,'T1',7X,'T2',7X,'FMEAN',7X,'VAR')
420=      WRITE(*,10)
430=      WRITE(*,110) T1,T2,FMEAN,VAR
440=  110 FORMAT (7X,4(1X,F9.8))
450=      WRITE(*,10)
460=      STOP
470=      END
480=      FUNCTION FUNC(Y)
490=      REAL FY,Y
500=      FUNC=(EXP(Y)-1.0)/1.718
510=      FY=FUNC
520=      RETURN
530=      END
540=      SUBROUTINE RAND(R)
550=      REAL R,S,XM
560=      INTEGER IS
```

```
570=       DATA S,XM /.23979,823.53478/
580=       S=S*XM
590=       IS=S
600=       R=S-IS
610=       S=R+.8234617
620=       RETURN
630=       END
640=       FUNCTION FAX(X)
650=       FAX=X
660=       RETURN
670=       END
..

RUN,FTNG
     ************************************************
        RN            TERM1         TERM2         TERM3
     ************************************************
         .467170       .346608       .467170       .014535
         .879721       .820854       .879721       .003465
         .630365       .511232       .630365       .014193
         .276951       .185740       .276951       .009319
         .228132       .149157       .228132       .006237
         .024116       .014208       .024116       .000099
         .009957       .005825       .009957       .000017
         .349598       .243595       .349598       .011237
         .055349       .033125       .055349       .000494
         .731006       .626990       .731006       .010819
         .158201       .099768       .158201       .003414
         .433205       .315595       .433205       .013832
         .908567       .861913       .908567       .002177
         .386255       .274424       .386255       .012506
         .243985       .160841       .243985       .006913
         .079153       .047945       .079153       .000974
         .334626       .231326       .334626       .010671
         .725597       .620469       .725597       .011052
         .704083       .594873       .704083       .011927
         .986214       .978501       .986214       .000059
     ************************************************
         T1          T2          FMEAN         VAR
     ************************************************
      .35614943 .43061259 .42553684 .00166658
     ************************************************
     61300 CM STORAGE USED.
     0.130 CP SECONDS COMPILATION TIME.
    STOP
     15100 MAXIMUM EXECUTION FL.
     0.058 CP SECONDS EXECUTION TIME.
..
```

APPENDIX F

Computer Program Listing with the Results for the Example of Antithetic Variates

```
L,A


  100=       PROGRAM ANTHTC
  110=C      ***************************************************
  120=C      THIS PROGRAM IS AN EXAMPLE OF USING ANTITHETIC VARIABLES AS
  130=C        V.R.T
  140=C      ***************************************************
  150=       ALPHA=.45
  160=       WRITE(*,300)
  170=       DO 1 L=1,10
  180=       SUM1=0.0
  190=       SUM2=0.0
  200=       N=20
  210=       ALPHA=ALPHA+.05
  220=       DO 100 I=1,N
  230=       CALL RAND (R1)
  240=       TERM1=FUNC(R1)
  250=       R2=1-R1
  260=       TERM2=FAX(R2)
  270=       TERM3=ALPHA*(TERM1)+(1-ALPHA)*(TERM2)
  280=       SUM1=SUM1+TERM3
  290=   100 SUM2=SUM2+TERM3**2
  300=       EI=SUM1/N
  310=       EI2=EI**2
  320=       S2=(SUM2-N*EI2)/(N-1)
  330=       WRITE(*,200) EI,S2,ALPHA
  340=   200 FORMAT(10X,'THE ESTIMATE = ',F8.7,3X,'THE VARIANCE=',F8.7,'
A = '
  350=      1    ,F4.2)
  360=       WRITE(*,300)
  370=   300 FORMAT(10X,58('*'))
  380=     1 CONTINUE
  390=       END
  400=       FUNCTION FUNC(Y)
  410=       FUNC=(EXP(Y)-1.0)/1.7182818
  420=       RETURN
  430=       END
  440=       FUNCTION FAX(X)
  450=       FAX=X
  460=       RETURN
  470=       END
  480=       SUBROUTINE RAND(R)
  490=       DATA S,XM /.23978,823.53476/
  500=       S=S*XM
  510=       IS=S
  520=       R=S-IS
  530=       S=R+.8234617
  540=       RETURN
  550=       END
```

```
RUN,FTN5

     61300 CM STORAGE USED.
     0.121 CP SECONDS COMPILATION TIME.


        ************************************************************
        THE ESTIMATE = .4627392   THE VARIANCE=.0004216  A =  .50
        ************************************************************
        THE ESTIMATE = .4523036   THE VARIANCE=.0016324  A =  .55
        ************************************************************
        THE ESTIMATE = .4281094   THE VARIANCE=.0025942  A =  .60
        ************************************************************
        THE ESTIMATE = .4817804   THE VARIANCE=.0095206  A =  .65
        ************************************************************
        THE ESTIMATE = .4264898   THE VARIANCE=.0111132  A =  .70
        ************************************************************
        THE ESTIMATE = .4180123   THE VARIANCE=.0257596  A =  .75
        ************************************************************
        THE ESTIMATE = .4686817   THE VARIANCE=.0356525  A =  .80
        ************************************************************
        THE ESTIMATE = .4080911   THE VARIANCE=.0403216  A =  .85
        ************************************************************
        THE ESTIMATE = .3821863   THE VARIANCE=.0481605  A =  .90
        ************************************************************
        THE ESTIMATE = .4816249   THE VARIANCE=.0717330  A =  .95
        ************************************************************
     END  ANTHTC
     15100 MAXIMUM EXECUTION FL.
     0.067 CP SECONDS EXECUTION TIME.
```

Vita

Mohamed Refat Ali Elhefny was born on 15 December 1947 at Minit Elkanh, Egypt. After graduation from high (secondary) school in July 1966, he attended the Military Technical College in Cairo, from which he received his Bachelor Degree in Electrical Engineering in May 1971. He worked in the field of maintenance, repair and design of electronic equipment in the Egyptian Army. In November 1977, he entered a graduate program in the Military Technical College, Cairo, after which he was awarded his degree (High Diploma) in Operations Research in March 1980. Upon graduation, he worked in one of the Operations Research Centers in the Egyptian Army until entering the School of Engineering, Air Force Institute of Technology in May 1982.

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Variance Reduction Techniques - Importance Sampling, Russian Roulette and Splitting, Systematic Sampling, Stratified Sampling, Antitheric Variables, Control Variates

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Title: Variance Reduction Techniques with applications.

Basic idea, theoritical foundation, procedure of implementation, fields of application and other specific characteristics of each of the known variance reduction techniques are illustrated. Steps needed to select the suitable variance reduction technique to a specific simulation problem are presented.

Examples are given to show how to implement the commonly used variance reduction techniques. Computer programs were written to perform those examples.

Results were interpreted and used to compare the efficiencies of different techniques.

Three studies in the field of inventory, queuing and computer performance measurement were discussed where various variance reduction techniques were employed. (see back page)

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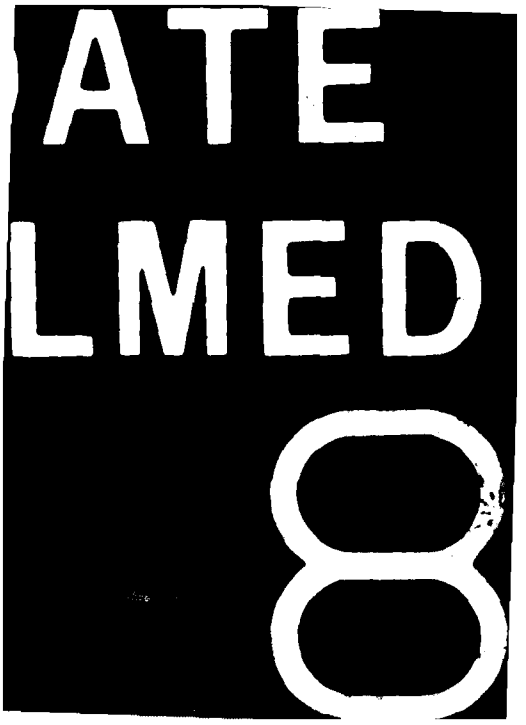
Conclusions and recommendations were given.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